# EXISTENCE OF $C^k$-INVARIANT FOLIATIONS FOR LORENZ-TYPE MAPS

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Abstract. In this paper under similar conditions to that Shaskov and Shil'nikov [1994] we show that a $C^{k+1}$ Lorenz-type map $T$ has a $C^k$ foliation which is invariant under $T$. This allows us to associate $T$ to a $C^k$ one-dimensional transformation.

Contents



## 1. Introduction

The **geometric Lorenz model** is an important example in dynamical systems, which was initially studied by Guckenheimer and Williams [Guc76, GW79, Wil79] and Afraimovich, Bykov and Shil'nikov [ABS77]. Their aim was to construct a simple mechanism which can give similar results to that Lorenz system

$$(\dot{x}, \dot{y}, \dot{z}) = (10(y-x), 28x - y - xz, -\frac{8}{3}z + xy),$$

introduced by Edward Lorenz [Lor63]. In this system, Edward Lorenz numerically found that most solutions tended to a certain attracting set, so-called *Lorenz Attractor* or *"strange" attractor*, and in so doing, he produced an important early example of " chaos ". Another fact that was noted by Lorenz: the *Lorenz Attractor* has *sensitivity to initial conditions* (*the butterfly effect*). No matter how close two solutions start, they will have a quite different

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behaviour in the future. The **geometric Lorenz model** has been analysed topologically and proved to possess a "strange" attractor with sensitive dependence on initial conditions. From these facts, we know that the **geometric Lorenz model** is crucial in the study of dynamical systems. For more details, see Viana [Via00].

Given a $C^{k+1}$ geometric Lorenz flow $X$ on $\mathbb{R}^{n+2}$, by definition there exists a $C^{k+1}$ Poincaré map $T_{_X} : D^* \to D$ , often so-called Lorenz-type map [ABS77]. In Shaskov and Shil'nikov[SS94] the authors showed that if a $C^2$ Lorenz-type map $T_{_X}$ satisfies certain conditions, then there exists a $C^1$ foliation which is invariant under $T_{_X}$. It allows us to associate $T_{_X}$ to a $C^1$ one dimensional Lorenz like map $f_{_X} : [a,b] \setminus \{c\} \to [a,b]$ . This association is so-called the reduction transformation $\mathcal{R}$, so we have $\mathcal{R}T_{_X} = f_{_X}$. This result allows $T_{_X}$ to be described in terms of a $C^1$ one-dimensional map.

Since the most deep results in one-dimensional dynamics (as the phase-parameter relations in Jakobson's Theorem [Jak81] and renormalization theory) relays on the study of sufficiently smooth families of transformations, to transfer this result to geometric Lorenz flow we need to study the smoothness of the reduction transformation $\mathcal{R}$. There are already impressive results using this approach ( see Rychlik [Ryc90], Rovella [Rov93], Morales, Pacifico and Pujals [MPP00], , Araújo and Varandas [AV12] ), however if we had a more deep knowledge of the regularity of $\mathcal{R}$ then far more significative results could be achieved.

In this work we extend the main result of Shil'nikov and Shaskov [SS94, Theorem] as well as [Ryc90, Corollary 4.2] of Rychlik, [Rov93, Proposition, p. 241] of Rovella and [MPP00, Lema 2] of Morales, Pacifico and Pujals . That is, we show that if a $C^{k+1}$ Lorenz type map $T$ satisfies certain conditions (see Assumption 2.2), then there exists a $C^k$ foliation which is invariant under $T_{_X}$. This theorem allows us to introduce new coordinates $\{(x,\eta)\}$ in $D$ such that the map $T_X$ has the form $\overline{T}_{_X}(x,\eta) = (\overline{F}_{_X}(x,\eta), \overline{G}_{_X}(\eta))$ (see Afraimovich and Pesin [AP87, P. 178]); where $\overline{F}_{_X}$ and $\overline{G}_{_X}$ are $C^k$ functions, so $T_{_X}$ can be associate to a $C^k$ one-dimensional transformation $\overline{G}_{_X} : [a,b] \setminus \{c\} \to [a,b]$. This association would allow us to study the dynamical properties of the original flow using powerful techniques of $C^k$ one-dimensional dynamics. Moreover, the result of this work can be useful in studying maps considered in Robinson [Rob84], Rychlik [Ryc90], Rovella [Rov93], Morales, Pacifico and Pujals [MPP00], Araújo and Varandas [AV12], Araujo and Pacifico [AP10] and in some other cases.

## 2. Statement of the Main Result

Let $\mathbb{R}^{n+1} := \mathbb{R}^n \times \mathbb{R}$ be a $(n+1)$-Euclidean space . From now on, the symbol $\| \, . \, \|$ denotes a norm in $\mathbb{R}^n$, if applied to a vector or for the corresponding matrix norm if applied to a matrix. We also use the notation

$$\|.\|_D = \sup_{(x,y)\in D^*} \|.\|$$

for norms of matrices and vector functions on $D^*$.

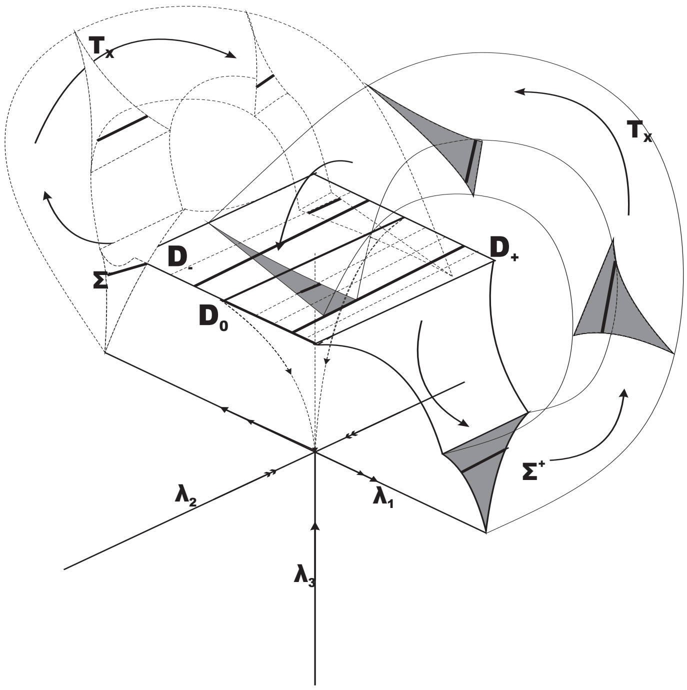


FIGURE 1. Geometric Lorenz flow in $\mathbb{R}^3$.

Define

$$\begin{aligned} D &:= \{(x,y) \in \mathbb{R}^{n+1} : \|x\| \leqslant 1, |y| \leqslant 1\}, \\ D_+ &:= \{(x,y) \in D : y > 0\}, \\ D_- &:= \{(x,y) \in D : y < 0\}, \\ D_0 &:= \{(x,y) \in D : y = 0\}, \\ D^* &:= D_- \cup D_+ = D \setminus D_0. \end{aligned} \tag{1}$$

Notice that the sets $D_+$ and $D_-$ are separate by the hyperplane $D_0$.

Let us consider the map $T : D^* \to D$ given by

$$T(x,y) = (F(x,y), G(x,y)) = (\overline{x}, \overline{y}), \tag{2}$$

where the vector function $F$ and the scalar function $G$ are differentiables on $D^*$ and $\partial_y G(x,y)$ is non-vanishing on $D^*$.

**Definition 2.1.** *We define the following functions:*

$$\begin{aligned} A(x,y) &:= \partial_x F(x,y)(\partial_y G(x,y))^{-1}, \\ B(x,y) &:= \partial_y F(x,y)(\partial_y G(x,y))^{-1}, \\ C(x,y) &:= \partial_x G(x,y)(\partial_y G(x,y))^{-1}. \end{aligned}$$

Here $A(x,y)$ is a $n \times n$ matrix, $B(x,y)$ is a $n$-column vector and $C(x,y)$ is a $n$-row vector.

**Assumption 2.2.** *We assume the following conditions hold on $T$:*

$(L_1)$ *The functions $F$ and $G$ have the forms*

$$F(x,y)=\begin{cases} x_+^* + |y|^\alpha[B_+^* + \varphi_+(x,y)], & y>0,\\ x_-^* + |y|^\alpha[B_-^* + \varphi_-(x,y)], & y<0,\end{cases}$$

$$G(x,y)=\begin{cases} y_+^* + |y|^\alpha[A_+^* + \psi_+(x,y)], & y>0,\\ y_-^* + |y|^\alpha[A_-^* + \psi_-(x,y)], & y<0,\end{cases}$$

*in a neighborhood of $D_0$, where $A_+^*, A_-^*$ are nonzero constants; $\alpha$ represents a strictly positive constant and the functions $\varphi_\pm$, $\psi_\pm$ are of class $C^{k+1}$. The derivatives of $\varphi_\pm$ and $\psi_\pm$ are uniformly bounded with respect to $x$ and satisfy the estimates:*

$$\left\|\frac{\partial^{l+m}\varphi_\pm(x,y)}{\partial x^l\partial y^m}\right\| \leqslant K|y|^{\gamma-m}, \quad \left\|\frac{\partial^{l+m}\psi_\pm(x,y)}{\partial x^l\partial y^m}\right\| \leqslant K|y|^{\gamma-m},$$

*where $\gamma > k-1$, $K$ is a positive constant, $l=0,1,\dots,k+1$ $m=0,1,\dots,k+1$ and $l+m\leqslant k+1$.*

$(L_2)$ *The following inequality holds:*

$$1-\|A\|_D > 2\sqrt{\|B\|_D\|C\|_D}.$$

$(L_3)$ *The following relations hold:*

(a)

$$\frac{(2!)^2\,(||A||_D+||C||_D||B||_D)\max\limits_{m+n=1}\{(||A||_D+||B||_D)^m(||C||_D+1)^n\}}{(||\partial_y G||_D)^{-1}\left(1+||A||_D+\sqrt{(1-||A||_D)^2-4||B||_D||C||_D}\right)^2}<1.$$

(b) *for $k\geqslant 2$*

$$\frac{(2k!)^2\,(||A||_D+||C||_D||B||_D)\max\limits_{m+n=k}\{(||A||_D+||B||_D)^m(||C||_D+1)^n\}}{(||\partial_y G||_D)^{-k}\left(1+||A||_D+\sqrt{(1-||A||_D)^2-4||B||_D||C||_D}\right)^2}<1,$$

*and*

$$\|\partial_y G\|_D \geqslant \frac{1}{4} \quad \text{or} \quad \|\partial_x F\|_D \geqslant \frac{1}{4}.$$

The following set will be useful for defining the domains of several maps:

$$D_x := \{x\in\mathbb{R}^n \quad \text{for which there exists a } y\in\mathbb{R} \quad \text{with} \quad (x,y)\in D\}.$$

Given a map $h: U\subset\mathbb{R}^n\to\mathbb{R}$, we defined its graph as

$$\mathrm{graph}(h) := \{(x,h(x)) : x\in U\}.$$

**Definition 2.3.** *A family of functions $\mathcal{F}_D=\{h(x)\}$ is called a foliation of $D$ with $C^m$ leaves ($m\geqslant 0$) given by the graphs of functions $y=h(x)$ if the following three conditions are satisfied:*

(a) *The domain $Dom(h(x))$ of every function $h(x)\in\mathcal{F}_D$ is an open and connected set in $D_x$ and its graph lies entirely in $D$;*

(b) *for every point $(x_0, y_0) \in D$ there is a unique function $h(x) \in \mathcal{F}_D$ such that $x_0 \in Dom(h(x))$ and $y_0 = h(x_0)$, this function will be denoted by $h(x; x_0, y_0)$;*
(c) *for every point $(x_0, y_0) \in D$ the function $x \mapsto h(x; x_0, y_0)$ is of class $C^m$.*

The graphs of the functions $h(x)$ are called the leaves of $\mathcal{F}_D$ and the leaf that contain $(x_0, y_0) \in D$ will be denoted by $\mathcal{F}_{(x_0,y_0)}$.

**Definition 2.4.** *A foliation $\mathcal{F}_D$ is called $C^r$-foliation $(r \geqslant 0)$ if the function*

$$(x; x_0, y_0) \mapsto h(x; x_0, y_0)$$

*is of class $C^r$.*

**Definition 2.5.** *A foliation $\mathcal{F}_D$ is called $T$-invariant if*

(a) *the hyperplane $D_0 \in \mathcal{F}_D$;*
(b) *for each leaf $\mathcal{F}_{(x_0,y_0)} \in \mathcal{F}_D$, with $\mathcal{F}_{(x_0,y_0)} \neq D_0$, there is $\mathcal{F}_{T(x_0,y_0)} \in \mathcal{F}$ such that $T(\mathcal{F}_{(x_0,y_0)}) \subset \mathcal{F}_{T(x_0,y_0)}$.*

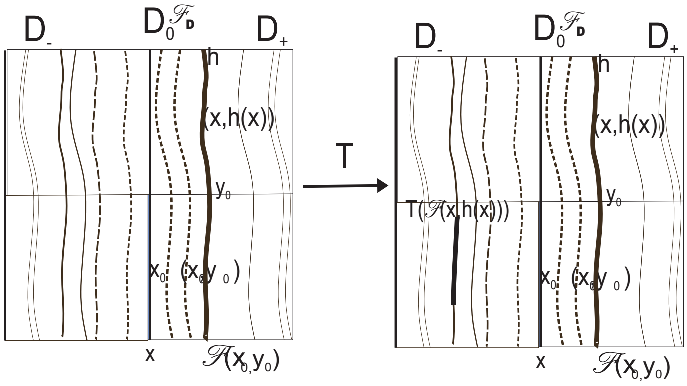


Figure 2. Geometric interpretation of a $T$-invariant foliation.

**Remark 2.6.** *Suppose that $\overline{\nu} : D \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a $C^k$ function completely integrable, that is, there exists a solution for the initial value problem for the differential equation*

$$\nabla y(x) = \overline{\nu}(x, y(x)), y(x_0) = y_0, \tag{3}$$

*for all $(x_0, y_0) \in D$, where $y : U(x_0) \subset D \to [-1, 1]$ and $U(x_0)$ is a neighborhood of $x_0$. Then, by using Frobenious-Dieudonné Theorem* [Die69, Theorem 10.9.5] *we have that*

$$\overline{\mathcal{F}}_D := \{graph(y) : \quad h \quad satisfing\ (3)\},$$

*determines a foliation, that is, the leaves are the graphs of the solutions of the differential equation defined by the function $\overline{\nu} : D \to \mathbb{R}^n$.*

We are ready to state our main result.

**Theorem 2.7** (**Main Theorem**). *Suppose that the map $T$ satisfies Assumption 2.2. Then, there is a $T$-invariant $C^k$-foliation $\mathcal{F}_D$ with $C^{k+1}$ leaves.*

As a byproduct of the preceding theorem we also have the following useful corollary, that say us that if the map $T$ satisfies Assumption 2.2 it can be introduced new coordinates $\{(x,\eta)\}$ in $D$ such that the map $T$ has the form of skew-product $\overline{T}(x,\eta) = (\overline{F}(x,\eta), \overline{G}(\eta))$; where $\overline{F}$ and $\overline{G}$ are $C^k$ functions, so $T$ can be associate to a one-dimensional transformation $\overline{G} : [a,b] \setminus \{c\} \to [a,b]$ of class $C^k$.

**Corollary 2.8.** *Suppose that the map $T$ satisfies Assumption 2.2. Then, there exists a change of variable $\chi : D \to D$ such that $T$ can be associate with a skew-product $\overline{T} : D^* \to D$ of class $C^k$ such that the diagram*

$$\begin{tikzcd} D^* \arrow[r, "T"] \arrow[d, "\chi"] & D \arrow[d, "\chi"] \\ D^* \arrow[r, "\overline{T}"] & D \end{tikzcd}$$

*is commutative, that is, $\chi \circ T = \overline{T} \circ \chi$ on $D^*$.*

*Proof.* The details can be found in [AP87, P. 178]. ∎

## 3. Overview of the Proof of Main Theorem 2.7

The principal aim of this section is to sketch the proof of our main theorem.

3.1. **The big picture.** Bearing in mind the Remark 2.6 and following the ideas of Robinson [Rob81]. The foliation $\mathcal{F}_D$ of Theorem 2.7 will be obtained as the integral surfaces of a $C^k$ completely integrable function $\nu : D \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$, which will be a fixed point of an appropriate graph transform $\Gamma$. Next, will be given a brief outline of the idea behind the graph transform $\Gamma$, which is also illustrated in Figure 3. Our goal is to find a $C^k$ integrable function $\nu^* : D \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$, so that for every integral surface $h$ its graphs is invariant under $T(x,y) = (F(x,y), G(x,y)) := (\overline{x}, \overline{y})$, which means that

$$\begin{aligned} F(x, h(x)) &= \overline{x}, \\ G(x, h(x)) &= \overline{h}(\overline{x}), \end{aligned}$$

where $\overline{h}$ is an integral surface of $\nu^*$. To find $\nu^*$ we take any completely integrable function $\overline{\nu} : D \to \mathbb{R}^n$ and seek a completely integrable function $\nu : D \to \mathbb{R}^n$ so that

$$\begin{aligned} F(x, h(x)) &= \overline{x}, \\ G(x, h(x)) &= \overline{h}(\overline{x}), \end{aligned}$$

where $h$ is an integral surface of $\nu$ and $\overline{h}$ is an integral surface of $\overline{\nu}$.
If such a function exists, we define the graph transform of $\overline{\nu}$ via $\Gamma(\overline{\nu}) := \nu$ and note that the desired function $\nu^*$ is a fixed point of the graph transform so that $\Gamma(\nu^*) = \nu^*$. It is not difficult to see that

$$\Gamma(\overline{\nu})(x,y) = \begin{cases} \dfrac{\overline{\nu} \circ T(x,y)\partial_y G(x,y) - \partial_y F(x,y)}{\partial_x F(x,y) - \overline{\nu} \circ T(x,y)\partial_x G(x,y)}, & y \neq 0, \\ 0, & y = 0. \end{cases}$$

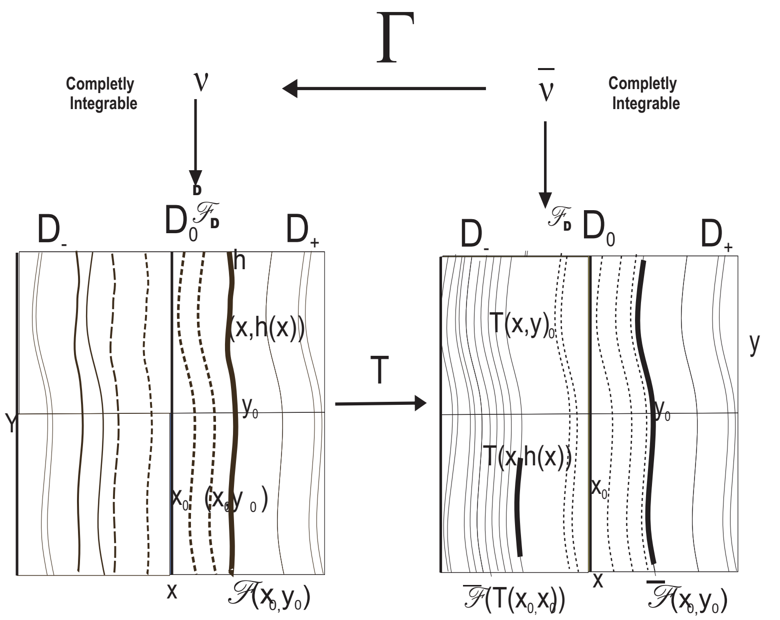


FIGURE 3. Graph transform Γ.

Notice that, in view of Definition 2.1, we can rewrite the operator Γ in the following way:

$$\Gamma(\overline{\nu})(x,y)=\begin{cases}\dfrac{(\overline{\nu}\circ TA-C)}{(1-\overline{\nu}\circ TB)}(x,y), & y\neq 0,\\ 0, & y=0.\end{cases}$$

It is not difficult to show that the graph transform Γ is well defined on a complete sub-space $\mathcal{A}_L$ of the continuous function from $D$ to $\mathbb{R}^n$, and that Γ has a fixed point $\nu^*$ (see Theorem 3.4). Our goal in this work is to show that the fixed point $\nu^*$ is a $C^k$ function completely integrable. Then, by using the Idea 1 we have that the graphs of the integral surfaces give the foliation $\mathcal{F}_D$ of Theorem 2.7.

3.2. **The Operator Γ.** Our goal in this section is to give a rigorous definition and state some properties of the operator Γ described informally in the last subsection. We begin by introducing the following definition.

**Definition 3.1.** *Let $L \geqslant 0$. We denote by $\mathcal{A}_L$ the set of all the functions $\nu : D \to \mathbb{R}^n$ which satisfies the following conditions:*

(*a*) *$\nu$ is continuous on $D$;*
(*b*) *$\|\nu\| \leqslant L$;*
(*c*) *$\nu(x,0)=0$, if $\|x\| \leqslant 1$.*

**Remark 3.2.** *Since $\mathbb{R}^n$ is a complete normed space, it is not difficult to show that $\mathcal{A}_L$ is a complete metric space with the norm of the supremum.*

Now we are ready to define the most important operator of our work. This operator is denoted by Γ and is defined as in [SS94, Eq. (6)] by

**Definition 3.3.**

$$\begin{array}{rcccl} \Gamma & : & \mathcal{A}_L & \longrightarrow & \Gamma(\mathcal{A}_L) \\ & & \overline{\nu} & \longmapsto & \nu = \Gamma(\overline{\nu}), \end{array} \tag{4}$$

*where the function* $\Gamma(\overline{\nu}) : D \to \mathbb{R}^{1\times n}$ *is given by*

$$\Gamma(\overline{\nu})(x,y) = \begin{cases} \dfrac{(\overline{\nu}\circ TA - C)}{(1-\overline{\nu}\circ TB)}(x,y), & y \neq 0, \\ 0, & y = 0, \end{cases}$$

*with the functions* $A, B$ *and* $C$ *as in Definition 2.1.*

Next, we list a few basic properties of the operator $\Gamma$. Details may be found in [SS94, Lemma 1] or [Vid14, Proposition 3.17].

**Proposition 3.4.** *There is a constant* $L \geqslant 0$ *such that*

(a) $\Gamma(\mathcal{A}_L) \subset \mathcal{A}_L$.
(b) *The operator* $\Gamma : \mathcal{A}_L \to \mathcal{A}_L$ *is a contraction.*
(c) *The operator* $\Gamma : \mathcal{A}_L \to \mathcal{A}_L$ *has a unique fixed point* $\nu^*$ *completely integrable function.*
(d) *The operator* $\Gamma$ *takes completely integrable function into completely integrable function. Moreover, if* $\overline{\mathcal{F}}_D$ *and* $\mathcal{F}_D$ *are foliations defined by the completely integrable functions* $\overline{\nu}$ *and* $\Gamma(\overline{\nu})$ *respectively; then* $T$ *takes every leaf* $\mathcal{F}_{(x_0,y_0)} \in \mathcal{F}_D$, $\mathcal{F}_{(x_0,y_0)} \neq D_0$, *into a part of the leaf* $\overline{\mathcal{F}}_{T(x_0,y_0)} \in \overline{\mathcal{F}}_D$, *that is,* $T(\mathcal{F}_{(x_0,y_0)}) \subset \overline{\mathcal{F}}_{T(x_0,y_0)}$.

**Remark 3.5.** *It is known from* [SS94, Eq. 8 1] *and* [Vid14, Eq. 3.47] *that* $L$ *can be taken as*

$$L = \frac{-(1-\|A\|) + \sqrt{(\|A\|-1)^2 - 4\|B\|\|C\|}}{2\|B\|}. \tag{5}$$

Let us begin stating the main proposition of this article.

**Proposition 3.6.** *Let* $L \geqslant 0$ *be as in Proposition 3.4. Then, the attracting fixed point* $\nu^*$ *of the operator* $\Gamma$ *is a function of class* $C^k$.

The proof of this proposition will be given with the following propositions which will be proven in the next sections.

**Proposition 3.7.** *If* $\mu \in \mathcal{A}_L$ *is a* $C^k$ *function. Then, the following statements hold:*

(a) $\lim_{(a,b)\to(x,0)} D^i(\Gamma(\mu))(a,b) = 0$, *for all* $1 \leqslant i \leqslant k$ *and* $(x,0) \in D_0$.
(b) *The function* $\Gamma(\mu) \in \mathcal{A}_L$ *is of class* $C^k$ *and* $D^i\Gamma(\mu)(x,0) = 0$, *for all* $1 \leqslant i \leqslant k$ *and* $(x,0) \in D_0$.

**Proposition 3.8.** *If* $\overline{\nu} \in \mathcal{A}_L$ *is a* $C^k$ *function and* $D^i\overline{\nu}(x,0) = 0$, *for all* $0 \leqslant i \leqslant k$ *and* $(x,0) \in D_0$. *Then, the following limit exists*

$$\lim_{n\to\infty} (\Gamma^n(\overline{\nu}), D(\Gamma^n(\overline{\nu})), \ldots, D^k(\Gamma^n(\overline{\nu}))) = (\nu^*, A_1, A_2, \ldots, A_k),$$

*where* $A_1, A_2, \ldots, A_k$ *are continuous functions.*

***Proof of Proposition 3.6***. Let $\nu \in \mathcal{A}_L$ be a $C^k$ function. By Proposition 3.6 we have that $\overline{\nu} := \Gamma(\nu)$ is a $C^k$ function and that $D^k\overline{\nu}(x,0) = 0$, From Proposition 3.8, we have that

$$\lim_{n\to\infty}(\Gamma^n(\overline{\nu}), D(\Gamma^n(\overline{\nu})), \ldots, D^k(\Gamma^n(\overline{\nu}))) = (\nu^*, A_1, A_2, \ldots, A_k).$$

Hence, and by using interchanging the order of differentiation and limit, see [Die69, Theorem 8.6.3], we obtain $D^j(\lim_{n\to\infty}\Gamma^n(\overline{\nu})) = A_j$, for $0 \leqslant j \leqslant k$. Thus, since $\nu^*$ is a global attracting fixed of $\Gamma$, it follows that $D^j(\nu^*) = A_j$, for $0 \leqslant j \leqslant k$. Therefore, since $A_j$ is a continuous function, it follows that the function $\nu^*$ is of class $C^k$, which concludes the proof of our main proposition.

∎

Now we are ready to prove our main Theorem 2.7.

***Proof of Theorem 2.7***. By Proposition 3.4(c) we have that the attracting fixed point $\nu^*$ of the operator $\Gamma$ is integrable and by Prop 3.6 we get that $\nu^*$ is of class $C^k$. Thus, the function $\nu^*$ defines a foliation $\mathcal{F}_D$ of class $C^k$ and by Proposition 3.4(d) it follows that the foliation $\mathcal{F}_D$ is $T$-invariant, which finishes the proof the of our main result. ∎

## 4. Proof of Proposition 3.7

The proof is somewhat lengthy, so we divide it into two parts. **In the first part:** we will establish a formula for the *kth order derivatives* of the function $\Gamma(\nu)$ at the points $(x, y)$, where the $y$-component stay away from zero . **In the second part:** we *estimate the norms of the ith derivatives* of the functions $A(x, y)$, $B(x, y)$ and $C(x, y)$, at the points $(x, y)$ around of a neighborhood of $D_0$.

4.1. **Part 1: Formula for Derivatives.** Before that, we introduce some definitions which will be useful in order to find suitable formulas. From now on, $L(E_1, \ldots, E_k; G)$ denotes the space of continuous $k$-multilinear maps of $E_1, \ldots, E_r$ to $G$. If $E_i = E, i \leqslant k$, this space is denoted $L^k(E, F)$. Moreover, $L^k_s(E; F)$ denotes the subspace of symmetric elements of $L^k(E, F)$.

**Definition 4.1** (Symmetrizing operator)**.** *The Symmetrizing operator $Sym^k$ is defined by*

$$\begin{array}{rcll} Sym^k : & L^k(E;F) & \longrightarrow & L^k(E;F) \\ & A & \longmapsto & Sym^k(A) = \dfrac{1}{k!}\displaystyle\sum_{\sigma\in S_k}\sigma A, \end{array}$$

*where $(\sigma A)(e_1, \ldots, e_k) = A(e_{\sigma(1)}, \ldots, e_{\sigma(k)})$ and $S_k$ is the group of permutations on $k$ elements.*

**Remark 4.2.** *The symmetrizing operator $Sym^k$ satisfies the following properties:*

(a) $Sym^k(L^k(E;F)) = L^k_s(E;F)$,
(b) $(Sym^k)^2 = Sym^k$,
(c) $\| Sym^k \| \leqslant 1$.

**Definition 4.3.** *Assume $\mathcal{B} \in L(F_1 \times F_2; G)$, we define the bilinear map.*

$$\begin{array}{rcccc} \phi^{(i,(k-i))} & : & L^i(E;F_1) \times L^{(k-i)}(E;F_2) & \longrightarrow & L^k(E;G) \\ & & (A_1, A_2) & \longmapsto & [\phi^{(i,(k-i))}(A_1,A_2)] \end{array}$$

*by*

$$[\phi^{(i,(k-i))}(A_1,A_2)](e_1,\ldots,e_k) = \mathcal{B}(A_1(e_1,\ldots e_i), A_2(e_{i+1},\ldots,e_k)). \tag{6}$$

**Definition 4.4.** *Let $\overline{\nu}_i : U \to L^i(E;F_1)$ and $\overline{\nu}_{(k-i)} : U \to L^{(k-i)}(E;F_2)$, we define*

$$\begin{array}{rcccc} \phi^{(i,(k-i))}(\overline{\nu}_i, \overline{\nu}_{(k-i)}) & : & U & \mapsto & L^k(E;G) \\ & & p & \to & \phi^{(i,(k-i))}(\overline{\nu}_i(p), \overline{\nu}_{(k-i)}(p)). \end{array} \tag{7}$$

**Definition 4.5.** *For every tuple $(q, r_1, r_2, \ldots, r_q)$, where $q > 1$, and $r_1 + \ldots + r_q = k$, we define the following continuous multilinear map*

$$\begin{array}{rl} \phi^{(q,r_1,\ldots,r_q)} & : L^q(F;G) \times L^{r_1}(E;F) \times \ldots \times L^{r_q}(E;F) \longrightarrow L^k(E;G) \\ & (\overline{v}_q, \overline{v}_{r_1}, \ldots, \overline{v}_{r_q}) \longmapsto \phi^{(q,r_1,\ldots,r_q)}(\overline{v}_q, \overline{v}_{r_1}, \ldots, \overline{v}_{r_q}), \end{array} \tag{8}$$

*where*

$$\phi^{(q,r_1,\ldots,r_q)}(\overline{v}_q, \overline{v}_{r_1}, \ldots, \overline{v}_{r_q}) : \underbrace{E \times \ldots \times E}_{k\text{-times}} \to G$$

*is defined as*

$$\begin{aligned} \phi^{(q,r_1,\ldots,r_q)}&(\overline{v}_q, \overline{v}_{r_1}, \ldots, \overline{v}_{r_q})(e_1,\ldots,e_k) \\ &= \overline{v}_q(\overline{v}_{r_1}(e_1,\ldots e_{j_{r_1}}), \ldots, \overline{v}_{r_q}(e_{(j_{r_1}+j_{r_2}+\ldots+j_{r_{(q-1)}}+1)}, \ldots, e_{(j_{r_1}+j_{r_2}+\ldots+j_{r_q})})). \end{aligned} \tag{9}$$

**Definition 4.6.** *Let $U \subset E$ such that $\overline{\nu}_{r_i} : U \to L^{r_i}(E;F), 1 \leqslant i \leqslant q$ and $f : V \subset U \to U$ are functions. Then, define*

$$\phi^{(q,r_1,\ldots,r_q)} * ((\overline{\nu}_q \circ f) \times \overline{\nu}_{r_1} \times \ldots \times \overline{\nu}_{r_q}) : U \to L^k(E;G) \tag{10}$$

*by*

$$u \to \phi^{(q,r_1,\ldots,r_q)}((\overline{\nu}_q \circ f(u)), \overline{\nu}_{r_1}(u), \ldots, \overline{\nu}_{r_q}(u)),$$

*where $\phi^{(q,r_1,\ldots,r_q)}((\overline{\nu}_q \circ f(u)), \overline{\nu}_{r_1}(u), \ldots, \overline{\nu}_{r_q}(u))$ as in Definition 4.5.*

Next, we define generalizations of the $kth$ derivative of the composition of two functions.

**Definition 4.7.** *Let $k_1 \geqslant k_3 \geqslant k_2 \geqslant 1$ be integers such that we have the functions $\overline{v}_q : V \subset F \to L^q(F,G)$, for $k_2 \leqslant q \leqslant k_3$ and that $f : U \to V$ is a function of class $C^{k_1-k_2+1}$, where $D^i f : U \to L^i(E,F), 0 \leqslant i \leqslant k_1 - k_2 + 1$ are the derivatives of $f$. Then, we define the function*

$$\mathcal{DC}^{(k_1,k_2,k_3)}((\overline{v}_{k_2}, \ldots, \overline{v}_{k_3}), f) : U \to L^{k_1}(E,F)$$

*given by*

$$
\begin{aligned}
(11)\quad & \mathcal{DC}^{(k_1,k_2,k_3)}((\overline{v}_{k_2},\ldots,\overline{v}_{k_3}),f)(p)\\
&:= Sym^{k_1}\left(\sum_{n=k_2}^{k_3}\sum_{r_1+\ldots+r_n=k_1}\frac{k_1!}{r_1!\ldots r_n!}\phi^{(n,r_1,\ldots,r_n)}*((\overline{v}_n\circ f)\times D^{r_1}f\times\ldots\times D^{r_n}f)\right)(p),
\end{aligned}
$$

*where* $\phi^{(n,r_1,\ldots,r_n)}*((\overline{v}_n\circ f)\times\overline{\nu}_{r_1}\times\ldots\times\overline{\nu}_{r_n})$ *as in Definition 4.6, and when* $k_1=k_3=0$ *and* $k_2=1$, *we define the function*

$$\mathcal{DC}^{(0,1,0)}(\overline{v}_0,f):U\to F$$

*given by*

$$\mathcal{DC}^{(0,1,0)}(\overline{v}_0,f)(p):=(\overline{v}_0\circ f)(p). \tag{12}$$

*Furthermore, if* $\overline{\nu}:V\subset F\to G$ *and* $f:U\subset E\to V\subset F$ *are* $C^{k_3}$, *then, will be used the following notation.*

$$\mathcal{DC}^{(k_1,k_2,k_3)}(\overline{\nu},f):=\mathcal{DC}^{(k_1,k_2,k_3)}(D^{k_2}(\overline{\nu}),\ldots,D^{k_3}(\overline{\nu}),f). \tag{13}$$

**Remark 4.8.** *If* $\overline{\nu}:D\to\mathbb{R}^{1\times n}$ *and* $T:D^*\to D$ *are functions* $C^k$, *then:*

(i) *on account of the chain rule applied to the function* $\overline{\nu}\circ T$ *it is possible to show that*

$$\mathcal{DC}^{(k,1,k)}(\overline{\nu},T):=D^k(\overline{\nu}\circ T).$$

*Therefore, we conclude that the function in Definitions 4.7 is a generalization of the* $k$*th derivative of the composite of two functions.*

(ii) *By Eq.* (13) *and the symmetry of the function* $D^k\overline{\nu}$ , *we obtain*

$$\mathcal{DC}^{(k,k,k)}(\overline{\nu},T):=k!(D^k\overline{\nu})\circ T\underbrace{DT\ldots DT}_{k-times}. \tag{14}$$

Next, we define generalizations of the $kth$ derivative of product of the map $(f\circ g)$ with $h$.

**Definition 4.9.** *Assume* $\mathcal{B}\in L(F_1\times F_2;G)$ *and that* $k_1\geqslant k_3\geqslant 1$; $k_2\geqslant 0$ *are integers such that* $f:U\subset E\to V\subset F$ *and* $B:U\to F_2$ *are functions of class* $C^{k_3}$ *and* $C^{k_1-k_2}$ *respectively, where* $D^iB:U\to L^i(E,F_2)$, *for* $0\leqslant i\leqslant k_1-k_2$ *are the derivatives of the function* $B$, *moreover consider the functions* $\overline{\nu}_i:V\subset F\to L^i(F,F_1)$, $0\leqslant i\leqslant k_3$. *Then, we define the map*

$$\mathcal{DCP}^{(k_1,k_2,k_3)}(f,(\overline{\nu}_0,\overline{\nu}_1,\ldots,\overline{\nu}_{k_3}),B):V\to L^{k_1}(E;G),$$

*given by*

$$
\begin{aligned}
(15)\quad & \mathcal{DCP}^{(k_1,k_2,k_3)}(f,(\overline{\nu}_0,\overline{\nu}_1,\ldots,\overline{\nu}_{k_3}),B)(p):=\\
& Sym^{k_1}\left(\sum_{n=k_2}^{k_3}\binom{k_1}{n}\phi^{(n,k_1-n)}\left(\mathcal{DC}^{(n,1,n)}\left((\overline{\nu}_1,\ldots,\overline{\nu}_n),f\right),D^{k_1-n}B\right)\right)(p),
\end{aligned}
$$

*where $\phi^{(n,k_1-n)}$ as in Definition 4.4. Moreover, if $\overline{\nu}: U \subset E \to F$ is a function of class $C^{k_3}$, then we will use the notation*

$$\mathcal{DCP}^{(k_1,k_2,k_3)}(\overline{\nu}, f, B) := \mathcal{DCP}^{(k_1,k_2,k_3)}(f, (\overline{\nu}, D(\overline{\nu}), \ldots, D^{k_3}(\overline{\nu})), B). \tag{16}$$

**Remark 4.10.** *Let $F_1$ and $F_2$ be the space of the $n$-columns and $n$-rows respectively. Then, we define the multilinear map $\mathcal{B}: F_1 \times F_2 \to \mathbb{R}$ given by $\mathcal{B}(A, B) = A \times B$, where $A \times B$ is the usual product of matrices. Assume that $\overline{\nu}: D \to \mathbb{R}^n$, $T: D^* \to D$ and $B: D^* \to F_1$ are $C^k$ functions. Then, by using Leibniz and chain rule applied to the functions $((\overline{\nu} \circ T).B)$ and $(\overline{\nu} \circ T)$, respectively and in view of Eq.* (4.8) *and Definition 4.9 it is easy to show*

$$\mathcal{DCP}^{(k,0,k)}(\overline{\nu}, T, B) := D^k((\nu \circ T)B). \tag{17}$$

*This show that the function in Definition 4.9 generalizes the $k$th derivative of the product of the map $(\nu \circ T)$ with $B$. This fact will be useful later.*

Next, we define generalizations of the $k$th derivative of the map $(1 - \nu \circ TB)^{-1}$, where $B$ is a function as in Definition 2.1 and $\nu \in \mathcal{A}_L$ is a function of class $C^k$.

**Definition 4.11.** *Let $(q, r_1, \ldots, r_q, r)$ be a tuple with $q \geqslant 1$, $r_1 + \ldots + r_q = k$ and $r = \max\{r_1, \ldots, r_q\}$ such that $\overline{\nu}_i : V \subset F \to L^i_s(F, F_1)$, $0 \leqslant i \leqslant r$ are functions and that $T: U \subset E \to V \subset F$ and $B: U \to F_2$ are functions of class $C^k$. Then, we define the map*

$$\prod^{(r_1,\ldots,r_q,r)} (\overline{\nu}_0, \ldots, \overline{\nu}_{r_q}, T, B) : U \to L^k(E, G)$$

*given by*

$$\prod^{(r_1,\ldots,r_q,r)} (\overline{\nu}_0, \ldots, \overline{\nu}_{r_q}, T, B) := \mathcal{DCP}^{(r_1,0,r_1)}(T, (\overline{\nu}_0, .., \overline{\nu}_{r_1}), B) \times .. \times \mathcal{DCP}^{(r_q,0,r_q)}((\overline{\nu}_0, .., \overline{\nu}_{r_q}), T, B), \tag{18}$$

*where $\mathcal{DCP}^{(r_i,0,r_i)}(T, (\overline{\nu}_0, .., \overline{\nu}_{r_i}), B)), 1 \leqslant i \leqslant q$ as in Definition 4.9.*

**Definition 4.12.** *Under the notations of Definition 4.11. Suppose that $k_1 \geqslant k_3 \geqslant k_2 \geqslant 1$ are integers . Then, we define the map*

$$\mathcal{DICP}^{(k_1,k_2,k_3)}(T, (\overline{\nu}_0, \ldots, \overline{\nu}_{(k_1-k_2)+1}), B) : U \to L^{k_1}(E, G)$$

*given by*

$$\begin{aligned} &\mathcal{DICP}^{(k_1,k_2,k_3)}((\overline{\nu}_0, \ldots, \overline{\nu}_{(k_1-k_2)+1}), T, B) \\ &:= Sym^{k_1}\left(\sum_{q=k_2}^{k_3} \sum_{r_1+\ldots+r_q=k_1} \frac{k!(-1)^q q!}{r_1! \ldots r_q!}(1 - \nu_0 \circ TB)^{-(q+1)} \prod^{(r_1,\ldots,r_q,r)} (\overline{\nu}_0, \ldots, \overline{\nu}_{r_q}, T, B)\right), \end{aligned} \tag{19}$$

*where $\prod^{(r_1,\ldots,r_q,r)} (\overline{\nu}_0, \ldots, \overline{\nu}_{r_q}, T, B)$ as in Definition 4.11. Furthermore, if $\overline{v}: U \subset F \to F_1$ is a $C^{k_1-k_2+1}$ map will be used the following notation*

$$\mathcal{DICP}^{(k_1,k_2,k_3)}(\overline{\nu}, T, B) := \mathcal{DICP}^{(k_1,k_2,k_3)}((\overline{\nu}, D(\overline{\nu}), \ldots, D^{k_1-k_2+1}(\overline{\nu})), T, B). \tag{20}$$

**Remark 4.13.** *Let $\nu \in \mathcal{A}_L, B, T$ be maps as in Definition 3.1, Definition 2.1 and Definition 2 respectively such that $\nu$ and $B$ are $C^k$. Define $\mathcal{I} : \mathbb{R} - \{0\} \to \mathbb{R}$ by $\mathcal{I}(x) = 1/x$. Since, the function $(1-\overline{\nu}\circ TB) : D^* \to \mathbb{R}$ is nonzero, it follows from chain rule applied to $\mathcal{I}\circ(1-\overline{\nu}\circ TB)$ and Definition 4.12 that*

$$\begin{aligned} D^k(1-\overline{\nu}\circ TB)^{-1} &= Sym^k \circ \sum_{q=1}^{k} \sum_{r_1+\ldots+r_q=k} \frac{k!q!}{r_1!\ldots r_q!} \frac{D^{r_1}(\overline{\nu}\circ TB)\ldots D^{r_q}(\overline{\nu}\circ TB)}{(1-\overline{\nu}\circ TB)^{(q+1)}} \\ &:= \mathcal{DICP}^{(k,1,k)}(\overline{\nu}, T, B). \end{aligned} \tag{21}$$

*This show that the function in Definition 4.12 generalizes the kth derivative of the map $(1-\nu\circ B)^{-1}$. This fact will be useful later.*

Now, using the last definitions we find one formula for the $kth$ derivative of the function $\Gamma(\nu)$ at the points $(x, y)$, with $y \neq 0$ (see Lema 4.18). This generalizes the formulas given in [SS94, Eq. (11)] and [MPP00, Eq. (42)], it is quite important to prove our main Proposition 3.6. We start by noticing the following simple but very useful lemma.

**Lemma 4.14.** *Under Definitions 2.1, 3.1 and* 3.3. *Assume that $\overline{\nu} \in \mathcal{A}_L$ is a $C^k$ function. Then, for $y \neq 0$ the following formulas hold:*

$$\begin{aligned} D(\Gamma(\overline{\nu}))(x,y) &= (\overline{\nu}\circ TA - C)D(1-\overline{\nu}\circ TB)^{-1}(x,y) \\ &\quad +D(\overline{\nu}\circ TA - C)(1-\overline{\nu}\circ TB)^{-1}(x,y) \\ &:= \left(U_1^1(\overline{\nu}, T, A, B, C) + U_2^1(\overline{\nu}, T, A, B, C)\right)(x,y); \end{aligned} \tag{22}$$

*for $k \geqslant 2$*

$$\begin{aligned} D^k(\Gamma(\overline{\nu}))(x,y) &= Sym^k \circ \overline{\nu}\circ TA - CD^k(1-\overline{\nu}\circ TB)^{-1}(x,y) \\ &+ Sym^k \circ D^k(\overline{\nu}\circ TA - C)(1-\overline{\nu}\circ TB)^{-1}(x,y) \\ &+ Sym^k \circ \sum_{q=1}^{k-1} \binom{k}{q} D^q(\overline{\nu}\circ TA - C)D^{k-q}(1-\overline{\nu}\circ TB)^{-1}(x,y). \\ &:= Sym^k \circ \left(U_1^k(\overline{\nu}, T, A, B, C) + U_2^k(\overline{\nu}, T, A, B, C)\right)(x,y) \\ &+ Sym^k \circ \left(U_3^k(\overline{\nu}, T, A, B, C)\right)(x,y). \end{aligned} \tag{23}$$

*Proof.* This is a direct consequence of Leibnitz's rule.

■

**Lemma 4.15.** *Under Definitions 2.1 and 3.1. Assume that $\overline{\nu} \in \mathcal{A}_L$ is a $C^k$ function and that $U_1^k(\overline{\nu}, T, A, B, C) : D^* \to L^k(\mathbb{R}^{n+1}, \mathbb{R}^n)$ is as in Lemma 4.14. Then the following formulas hold:*

$$U_1^1(\overline{\nu}, T, A, B, C) = (\overline{\nu}\circ TA - C)(1-\overline{\nu}\circ TB)^{-2}\left(\overline{\nu}\circ TDB + D\overline{\nu}\circ T.DT.B\right), \tag{24}$$

*for $k \geqslant 2$*

$$
\begin{aligned}
U_1^k(\overline{\nu},T,A,B,C) &= (\overline{\nu}\circ TA - C)k!(1-\overline{\nu}\circ TB)^{-2} Sym^k \circ \Big(\mathcal{DC}^{(k,k,k)}(\overline{\nu},T)B\Big)\\
&+ (\overline{\nu}\circ TA - C)k!(1-\overline{\nu}\circ TB)^{-2} Sym^k \circ \Big(\mathcal{DC}^{(k,1,(k-1))}(\overline{\nu},T)\Big)\\
&+ (\overline{\nu}\circ TA - C)k!(1-\overline{\nu}\circ TB)^{-2} Sym^k \circ \Big(\mathcal{DCP}^{(k,0,(k-1))}(\overline{\nu},T,B)\Big)\\
&+ (\overline{\nu}\circ TA - C)\mathcal{DICP}^{(k,2,k)}(\overline{\nu},T,B), \qquad (25)
\end{aligned}
$$

*where $\mathcal{DC}^{(k_1,k_2,k_3)}(\overline{\nu},T)$, $\mathcal{DCP}^{(k_1,k_2,k_3)}(\overline{\nu},T,B)$ and $\mathcal{DICP}^{(k_1,k_2,k_3)}(\overline{\nu},T,B))$ as in Definitions 4.7, 4.9 and 4.12, respectively.*

*Proof.* We do the proof for the formula (25). The proof of the formula (24) is straightforward. From assumption and Remark 4.13 it follows that

$$
\begin{aligned}
U_1^k(\overline{\nu},T,A,B,C) &= (\overline{\nu}\circ TA - C)D^k(1-\overline{\nu}\circ TB)^{-1}.\\
&:= (\overline{\nu}\circ TA - C)I_1^k. \qquad (26)
\end{aligned}
$$

We observe that, for $k \geqslant 2$, by the chain rule,

$$
\begin{aligned}
I_1^k &= k!(1-\overline{\nu}\circ TB)^{-2} Sym^k \circ \big(D^k(\overline{\nu}\circ TB)\big)\\
&+ Sym^k \circ \sum_{q=2}^{k} \sum_{r_1+\ldots+r_q=k} \frac{k!}{r_1!\ldots r_q!}(1-\overline{\nu}\circ TB)^{-(q+1)} D^{r_1}(\overline{\nu}\circ TB)\ldots D^{r_q}(\overline{\nu}\circ TB).
\end{aligned}
$$

$$
(27) := k!(1-\overline{\nu}\circ TB)^{-2} Sym^k \circ \big(I_2^k\big) + Sym^k \circ \sum_{q=2}^{k} \sum_{r_1+\ldots+r_q=k} \frac{k!}{r_1!\ldots r_q!}(1-\overline{\nu}\circ TB)^{-(q+1)} I_3^k.
$$

Now applying Leibniz's rule to the function $(\overline{\nu}\circ T)B$ , it follows that

$$
\begin{aligned}
I_2^k &= Sym^k \circ \big(D^k(\overline{\nu}\circ T)B\big) + Sym^k \circ \left(\sum_{q=0}^{k-1} \binom{k}{q} D^q(\overline{\nu}\circ T).D^{k-q}(B)\right). \qquad (28)\\
&:= Sym^k \circ \big(I_{2,1}^k B\big) + Sym^k \circ \left(\sum_{q=0}^{k-1} \binom{k}{q} I_{2,2}^k\right).
\end{aligned}
$$

Moreover, by using the chain rule to the functions $I_{2,1}^k$ and $I_{2,2}^k$ respectively, we get

$$
\begin{aligned}
I_{2,1}^k &= Sym^k \circ \left( k!(D^k\overline{\nu})\circ T \underbrace{DT\ldots DT}_{k-times} \right)\\
&+ Sym^k \circ \left( \sum_{q=1}^{k-1} \sum_{r_1+\ldots+r_q=k} \frac{k!}{r_1!\ldots r_q!}(D^q\overline{\nu})\circ T.D^{r_1}T\ldots D^{r_q}T \right), \qquad (29)
\end{aligned}
$$

and

$$
(30) \qquad I_{2,2}^k = Sym^q \circ \left( \sum_{n=1}^{q} \sum_{r_1+\ldots+r_n=q} \frac{q!}{r_1!\ldots r_n!}(D^n\nu)\circ T.D^{r_1}T\ldots D^{r_n}T \right).D^{k-q}(B).
$$

Therefore, by replacing (30) and (29) into (28), and using that $Sym^k \circ Sym^k = Sym^k$, we get

$$\begin{aligned} I_2^k &= Sym^k\left(k!(D^k\overline{\nu})\circ T\underbrace{DT\dots DT}_{k-times}B\right)\\ &+ Sym^k\circ\left(\sum_{q=1}^{k-1}\sum_{r_1+\ldots+r_q=k}\frac{k!}{r_1!\dots r_q!}(D^q\overline{\nu})\circ F.D^{r_1}T\dots D^{r_q}TB\right)\\ &+ Sym^k\circ\sum_{q=0}^{k-1}\binom{k}{q}Sym^q\left(\sum_{n=1}^{q}\sum_{r_1+\ldots+r_n=q}\frac{q!(D^n\nu)\circ T.D^{r_1}T\dots D^{r_n}T}{r_1!\dots r_n!}.D^{k-q}(B)\right). \end{aligned} \tag{31}$$

Hence, on account of Definitions 4.7 and 4.9 we have

$$I_2^k := Sym^k(\mathcal{DC}^{(k,1,k)}(\overline{\nu},T)B) + \mathcal{DC}^{(k,1,(k-1))}(\overline{\nu},T,B) + \mathcal{DCP}^{(k,0,(k-1))}(\overline{\nu},T,B). \tag{32}$$

By similar computation as above, in view of Definitions 4.7 and 4.9 we reach that

$$I_3^k := \mathcal{DCP}^{(r_1,0,r_1)}(\overline{\nu},T,B)\dots\mathcal{DCP}^{(r_q,0,r_q)}(\overline{\nu},T,B). \tag{33}$$

Hence, by using Definition 4.9, Eq. (33) becomes

$$I_3^k = \prod^{(r_1,\ldots,r_q,r)}(\overline{\nu},T,B). \tag{34}$$

Whence, on account of Definition 4.12, we get

$$Sym^k\circ\left(\sum_{q=2}^{k}\sum_{r_1+\ldots+r_q=k}\frac{k!}{r_1!\dots r_q!}(1-\overline{\nu}\circ TB)^{-(q+1)}I_3^k\right) := \mathcal{DICP}^{(k,2,k)}(\overline{\nu},T,B). \tag{35}$$

Thus, by replacing (35) and (32) into (27), we get

$$\begin{aligned} I_1^k &= Sym^k\circ\Big(k!(1-\overline{\nu}\circ TB)^{-2}(\mathcal{DC}^{(k,1,k)}((\overline{\nu},T)B))\Big)\\ &+ Sym^k\circ\Big(k!(1-\overline{\nu}\circ TB)^{-2}(\mathcal{DC}^{(k,1,(k-1))}(\overline{\nu},T,B))\Big)\\ &+ Sym^k\circ\Big(k!(1-\overline{\nu}\circ TB)^{-2}(\mathcal{DCP}^{(k,0,(k-1))}(\overline{\nu},T,B))\Big)\\ &+ \mathcal{DICP}^{(k,2,k)}(\overline{\nu},T,B). \end{aligned} \tag{36}$$

Therefore, by replacing (36) into (26), and on account of $Sym^k \circ Sym^k = Sym^k$, it follows formula (25). ∎

**Lemma 4.16.** *Under Definitions 2.1 and 3.1. Assume that $\overline{\nu} \in \mathcal{A}_L$ is a $C^k$ function and that $U_2^k(\overline{\nu},T,A,B,C) : D^* \to L^k(\mathbb{R}^{n+1},\mathbb{R}^n)$ is as in Lemma 4.14. Then, the following formulas hold:*

$$U_2^1(\overline{\nu},T,A,B,C) = \left((D\overline{\nu})\circ TDTA + \overline{\nu}\circ TDA - D(C)\right)(1-\overline{\nu}\circ TB)^{-1}, \tag{37}$$

*for $k \geqslant 2$*

$$\begin{aligned} U_2^k(\overline{\nu}, T, A, B, C) &= Sym^k\left(\mathcal{DC}^{(k,k,k)}(\overline{\nu}, T)A\right)(1-\overline{\nu}\circ TB)^{-1} \\ &+ Sym^k\left(\mathcal{DC}^{(k,1,k-1)}(\overline{\nu}, T)A\right)(1-\overline{\nu}\circ TB)^{-2} \\ &+ Sym^k\left(\mathcal{DCP}^{(k,0,k-1)}(\overline{\nu}, T, A)\right)(1-\overline{\nu}\circ TB)^{-2} \\ &- Sym^k\left(D^k(C)\right).(1-\overline{\nu}\circ TB)^{-2}, \end{aligned} \tag{38}$$

*where $\mathcal{DC}^{(k_1,k_2,k_3)}(\overline{\nu}, T)$, $\mathcal{DCP}^{(k_1,k_2,k_3)}(\overline{\nu}, T, B)$, $\mathcal{DICP}^{(k_1,k_2,k_3)}(\overline{\nu}, T, B))$ as in Definitions 4.7, 4.9 and 4.12, respectively.*

*Proof.* The proof is quite similar to the development from Eq. (27). ■

**Lemma 4.17.** *Under Definitions 2.1 and 3.1. Assume that $\overline{\nu} \in \mathcal{A}_L$ is a $C^k$ function and that $U_3^k(\overline{\nu}, T, A, B, C) : D^* \to L^k(\mathbb{R}^{n+1}, \mathbb{R}^n)$ as in Lemma 4.14. Then the following equality holds:*

$$\begin{aligned} &U_3^k(\overline{\nu}, T, A, B, C) = \\ &\quad Sym^k\left(\sum_{q=1}^{k-1}\binom{k}{q}\phi^{(q,k-q)}((\mathcal{DCP}^{(q,0,q)}(\overline{\nu}, T, A) - D^q(C)), \mathcal{DICP}^{(k-q,1,k-q)}(\overline{\nu}, T, B))\right), \end{aligned} \tag{39}$$

*where $\mathcal{DCP}^{(k_1,k_2,k_3)}(\overline{\nu}, T, B)$, $\phi^{(q,k-q)}$, $\mathcal{DICP}^{(k_1,k_2,k_3)}(\overline{\nu}, T, B))$ as in Definitions 4.9, 4.6 and 4.12, respectively.*

*Proof.* The proof is similar to that of Lemma 4.15. For more details, see all the developments of the formulas $I_1^k$ (Eq. (27)) and $I_2^k$ (Eq. (28)). ■

As a direct consequence of Lemmas 4.14, 4.15, 4.16 and 4.17 we obtain a formula for the *kth* derivative of the function $\Gamma(\overline{\nu})$ at the point $(x, y)$, for $y \neq 0$.

**Lemma 4.18.** *Under Definitions 2.1, 3.1 and* 3.3. *Assume that $\overline{\nu} \in \mathcal{A}_L$ is a $C^k$ function a that $y \neq 0$, then the following formulas hold:*

$$\begin{aligned} D(\Gamma(\overline{\nu}))(x,y) = \big(&(\overline{\nu}\circ TA - C)(1-\overline{\nu}\circ TB)^{-2}(\overline{\nu}\circ TDB + D\overline{\nu}\circ T.DT.B) \\ &+ (\overline{\nu}\circ TDTA + \overline{\nu}\circ TDA - DC)(1-\overline{\nu}\circ TB)^{-1}\big)(x,y). \end{aligned} \tag{40}$$

*for* $k \geqslant 2$

$$
\begin{aligned}
D^k(\Gamma(\overline{\nu}))(x,y) &= \Big((\overline{\nu}\circ TA - C)(1-\overline{\nu}\circ TB)^{-2} Sym^k \circ \Big(\mathcal{DC}^{(k,k,k)}(\overline{\nu},T)B\Big)\\
&+ (\overline{\nu}\circ TA - C)(1-\overline{\nu}\circ TB)^{-2} Sym^k \circ \Big(\mathcal{DC}^{(k,1,(k-1))}(\overline{\nu},T)\Big)\\
&+ (\overline{\nu}\circ TA - C)(1-\overline{\nu}\circ TB)^{-2} Sym^k \circ \Big(\mathcal{DCP}^{(k,0,(k-1))}(\overline{\nu},T,B)\Big)\\
&+ (\overline{\nu}\circ TA - C)\mathcal{DICP}^{(k,2,k)}(\overline{\nu},T,B)\\
&+ (1-\overline{\nu}\circ TB)^{-1} Sym^k \circ \Big(\mathcal{DC}^{(k,k,k)}(\overline{\nu},T)A\Big)\\
&+ (1-\overline{\nu}\circ TB)^{-2} Sym^k \Big(\mathcal{DC}^{(k,1,k-1)}(\overline{\nu},T)A + \mathcal{DCP}^{(k,0,k-1)}(\overline{\nu},T,A)\Big)\\
&- (1-\overline{\nu}\circ TB)^{-2} Sym^k \circ \big(D^k(C)\big) + U_3^k(\overline{\nu},T,A,B,C).
\end{aligned} \tag{41}
$$

*where* $\mathcal{DC}^{(k_1,k_2,k_3)}(\overline{\nu},T)$, $\mathcal{DCP}^{(k_1,k_2,k_3)}(\overline{\nu},T,B)$, $\mathcal{DICP}^{(k_1,k_2,k_3)}(\overline{\nu},T,B))$, $\phi^{(q,k-q)}$ *as in Definitions 4.7, 4.9, 4.12 and 4.6, respectively and* $U_3^k(\overline{\nu},T,A,B,C)$ *as in Lemma* 4.17.

4.2. **Part 2: The norm of the** *ith* **derivative.** In this sub-section will be estimated the norms of the *ith* derivative of the functions $A(x,y)$, $B(x,y)$ and $C(x,y)$ around of a neighborhood of $D_0$. We start by noticing the following simple but useful lemma.

**Lemma 4.19.** *Let*

$$
d(x,y) = \begin{cases} \alpha(A_+^* + \partial_x\psi_+(x,y)) + |y|\partial_y\psi_+(x,y), & y>0,\\[2ex] \alpha(A_-^* + \partial_x\psi_-(x,y)) + |y|\partial_y\psi_-(x,y), & y<0, \end{cases} \tag{42}
$$

*and*

$$
\rho(x,y) = \begin{cases} \dfrac{1}{\min\limits_{0\leqslant j\leqslant i} |\ (\alpha A_+^* + \psi_+(x,y) + y\partial_y\psi_+(x,y))\ |^{j+1}}, & y>0,\\[4ex] \dfrac{1}{\min\limits_{0\leqslant j\leqslant i} |\ (\alpha A_-^* + \psi_-(x,y) + |y|\partial_y\psi_-(x,y))\ |^{j+1}}, & y<0, \end{cases} \tag{43}
$$

*Then,* $d$ *and* $\rho$ *are defined in a neighborhood* $\widetilde{U}$ *of* $D_0$ *and there exists a constant* $C \geqslant 0$ *such that the following estimative holds:*

$$
\|D^i(d(x,y)^{-1})\| \leqslant C\rho(x,y)|y|^{\gamma-i}, \quad \textit{for all} \quad (x,y)\in\widetilde{U}. \tag{44}
$$

Moreover, the limit

$$
\lim_{(x,y)\to(a,0^{\pm})} \rho(x,y) \tag{45}
$$

exists, for all $(a,0)\in D_0$.

*Proof.* Since $A^*_{\pm} \neq 0$, then the estimate (44) is a directly consequence of Example 4.13 and norm properties. From Assumption 2.2($L1$), it follows that the limit $\lim_{(x,y)\to(a,0^{\pm})} \rho(x,y)$ exists. This finishes the proof of lemma. ■

As a consequence of Lemma 4.19 and Leibnitz rule we get:

**Corollary 4.20.** *Let $0 \leqslant i \leqslant k$ be a integer. Assume $A, B$ and $C$ as in Definition 2.1 and $\rho$ as in Lemma 4.19. Then, there is a constant $C \geqslant 0$ such that the following inequalities hold:*

$$\| D^i A(x,y) \| \leqslant C\rho(x,y)|y|^{\gamma-i+1}, \tag{46}$$

$$\| D^i C(x,y) \| \leqslant C\rho(x,y)|y|^{\gamma-i+1}, \tag{47}$$

$$||D^i B(x,y)|| \leqslant C\rho(x,y)|y|^{\gamma-i}, \tag{48}$$

*in a neighborhood of $D_0$.*

**Corollary 4.21.** *Assume $T(x,y) = (F(x,y), G(x,y))$ is a map that satisfies Assumption 2.2($L_1$). Then, the following relation holds:*

$$\|D^k T(a,b)\| \leqslant const\|b\|^{\alpha-k}, \tag{49}$$

*in a neighborhood of $D_0$, where **const** denotes a positive constant.*

*Proof.* The proof is a direct consequence of Assumption 2.2(Eq. (2.2)) and Leibnitz rule. ■

**Lemma 4.22.** *Let $A, B$ and $C$ be as in Definition 2.1. Assume that $\overline{\nu} \in \mathcal{A}_L$ is a $C^k$ functions and that $U^k_1(\overline{\nu}, T, A, B, C)$, $U^k_2(\overline{\nu}, T, A, B, C)$ and $U^k_3(\overline{\nu}, T, A, B, C)$ as in Lemmas 4.15, 4.16 and 4.17, respectively . Then*

$$\lim_{(a,b)\to(x,0)} U^k_1(\overline{\nu}, T, A, B, C)(a,b) = 0, \tag{50}$$

$$\lim_{(a,b)\to(x,0)} U^k_2(\overline{\nu}, T, A, B, C)(a,b) = 0, \tag{51}$$

$$\lim_{(a,b)\to(x,0)} U^k_3(\overline{\nu}, T, A, B, C)(a,b) = 0, \tag{52}$$

*for every $(x,0) \in D_0$.*

*Proof.* The result is easy to prove for $k = 1$. We prove the result for the case $k \geqslant 2$.

By Lemma 4.15($k \geqslant 2$), Definition 3.1 and Remark 4.2(iii) we have

$$\begin{aligned}\|U_1^k(\overline{\nu}, T, A, B, C)\| &\leqslant \frac{(L\|A(a,b)\| + \|C(a,b)\|)}{(1 - L\|B\|)^2}\|k!\mathcal{DC}^{(k,k,k)}(\overline{\nu}, T)(a,b)\|\|B\| \\ &+ \frac{(L\|A(a,b)\| + \|C(a,b)\|)}{(1 - L\|B\|)^2}\|k!\mathcal{DC}^{(k,1,(k-1))}(\overline{\nu}, T)(a,b)\| \\ &+ \frac{(L\|A(a,b)\| + \|C(a,b)\|)}{(1 - L\|B\|)^2}\|\mathcal{DCP}^{(k,0,(k-1))}(\overline{\nu}, T, B)(a,b)\| \\ &+ (L\|A(a,b)\| + \|C(a,b)\|)\|\mathcal{DICP}^{(k,2,k)}(\overline{\nu}, T, B)(a,b)\|.\end{aligned} \tag{53}$$

To estimate the first expression of (53). From (14) and norm properties we have

$$\|\mathcal{DC}^{(k,k,k)}(\overline{\nu}, T)(a,b)\| \leqslant \|k!(D^k\overline{\nu}) \circ T\|\|DT(a,b)\|^k. \tag{54}$$

Since $\nu$ is of class $C^k$, and by using Corollary 4.21 we get

$$\|\mathcal{DC}^{(k,k,k)}(\overline{\nu}, T)(a,b)\| \leqslant const|b|^{\gamma-k}. \tag{55}$$

Whence, in view of Corollary 4.20 we get

$$\frac{(L\|A(a,b)\| + \|C(a,b)\|)}{(1 - L\|B\|)^2}\|k!\mathcal{DC}^{(k,k,k)}(\overline{\nu}, T)(a,b)\| \leqslant const\frac{|b|^{\alpha-k+\gamma+1}}{(1 - L\|B\|)^2}. \tag{56}$$

By similar arguments one can estimates remaining expressions of (53) to obtain

$$\frac{(L\|A(a,b)\| + \|C(a,b)\|)}{(1 - L\|B\|)^2}\|k!\mathcal{DC}^{(k,1,(k-1))}(\overline{\nu}, T)(a,b)\| \leqslant |b|^{\alpha-k+\gamma+1}(1 - L\|B\|)^2 const. \tag{57}$$

$$\frac{(L\|A(a,b)\| + \|C(a,b)\|)}{(1 - L\|B\|_D)^2}\|\mathcal{DCP}^{(k,0,(k-1))}(\overline{\nu}, T, B)(a,b)\| \leqslant \frac{|b|^{\gamma-k+1}}{(1 - L\|B\|_D)^2}. \tag{58}$$

$$(L\|A(a,b)\| + \|C(a,b)\|)\|\mathcal{DICP}^{(k,2,k)}(\overline{\nu}, T, B)(a,b)\| \leqslant const|b|^{\alpha-k+\gamma+1}. \tag{59}$$

Therefore, combining the four estimates (59), (58), (57) and (56) with (53) we obtain

$$\|U_1^k(\overline{\nu}, T, A, B, C)\| \leqslant const|b|^{\alpha-k+\gamma+1}. \tag{60}$$

Hence, since $\gamma > k-1$ and $\alpha > 0$ (see Assumption 2.2($L_1$)) we reach that

$$\lim_{(a,b)\to(x,0)} \|U_1^k(\overline{\nu}, T, A, B, C)(a,b)\| = 0.$$

Repeating the same procedure followed to deduce estimate (50) , we get estimates (51) and (52). Thus, we conclude the proof of corollary. ∎

***Proof of Proposition 3.7***. This is a direct consequence of Lemma 4.22. ∎

## 5. Proof of Proposition 3.8

The proof of Proposition 3.8 was influenced by the ideas contained in the articles [Rob81, p. 313] and [SS94, Eq. (3)]. The proof is quite long and technical, so we divide it into two steps. Before that, we give the following definition.

**Definition 5.1.** *We define the set $D_i$ of all the continuous functions $\nu_i : D \to L^i_s(\mathbb{R}^{n+1}, \mathbb{R}^n)$ such that $\nu_i(x, 0) = 0$, for all $(x, 0) \in D_0$ that is,*

$$\mathcal{D}_i := \{\nu_i : D \to L^i_s(\mathbb{R}^{n+1}, \mathbb{R}^n) : \nu_i(x, 0) = 0, \quad \textit{for all} \quad (x, 0) \in D_0; \quad v_i \textit{ is continuous}\},$$

*for every $1 \leqslant i \leqslant k$, and $\mathcal{D}_0 := \mathcal{A}_L$.*

The proof of Proposition 3.8 is somewhat lengthy, so we divide it into two parts. **In the first part:** we show the *existence of functions* $\Psi^i : \mathcal{D}_0 \times \mathcal{D}_1 \times \ldots \times \mathcal{D}_i \to D_i$, so that $D^i(\Gamma(\overline{\nu}_0)) = \Psi^i(\overline{\nu}_0, D(\overline{\nu}_0), \ldots, D^i(\overline{\nu}_0))$, for all $0 \leqslant i \leqslant k$. **In the second part:** we show that the function $\widetilde{N}_i : \mathcal{D}_0 \times \mathcal{D}_1 \times \ldots \times \mathcal{D}_i \to \mathcal{D}_0 \times \mathcal{D}_1 \times \ldots \times \mathcal{D}_i$ given by $\widetilde{N}_i(\overline{\nu}_0, \overline{\nu}_1, \ldots, \overline{\nu}_i) = (\Gamma(\overline{\nu}_0), \Psi^1(\overline{\nu}_0, \overline{\nu}_1), \ldots, \Psi^i(\overline{\nu}_0, \overline{\nu}_1, \ldots, \overline{\nu}_i))$ *have a global attracting fixed point* $(A_0, A_1, \ldots, A_i)$, for all $0 \leqslant i \leqslant k$.

### 5.1. Part 1: Defining the functions $\Psi^i$.

We start by defining a generalization of the function $U^k_1$ (see Corollary 4.15).

**Definition 5.2.** *Let $1 \leqslant i \leqslant k$ be a integer. Let $D_0$ and $\mathcal{D}_i$ be sets as in Definition 5.1. We define the function $U^i_1 : \mathcal{D}_0 \times \mathcal{D}_1 \times \ldots \times \mathcal{D}_i \to D_i$ given by*

$$U^1_1(\overline{\nu}_0, \overline{\nu}_1) = (\overline{\nu}_0 \circ TA - C)(1 - \overline{\nu}_0 \circ TB)^{-2} \left(\overline{\nu}_0 \circ TDB + \overline{\nu}_1 \circ T.DT.B\right), \tag{61}$$

*for $i \geqslant 2$*

$$\begin{aligned} U^i_1(\overline{\nu}_0, \ldots, \overline{\nu}_i) &= \frac{(\overline{\nu}_0 \circ TA - C)i!}{(1 - \overline{\nu}_0 \circ TB)^2} Sym^i \circ \mathcal{DC}^{(i,i,i)}(\overline{\nu}_i, T)B \\ &+ \frac{(\overline{\nu}_0 \circ TA - C)i!}{(1 - \overline{\nu}_0 \circ TB)^2} Sym^i \circ \mathcal{DC}^{(i,1,(i-1))}(\overline{\nu}_1, \ldots, \overline{\nu}_{(i-1)}, T) \\ &+ \frac{(\overline{\nu}_0 \circ TA - C)i!}{(1 - \overline{\nu}_0 \circ TB)^2} Sym^i \circ \mathcal{DCP}^{(i,0,(i-1))}(\overline{\nu}_0, \ldots, \overline{\nu}_{(i-1)}, T, B) \\ &+ (\overline{\nu}_0 \circ TA - C)\mathcal{DICP}^{(i,2,i)}(\overline{\nu}_0, \ldots, \overline{\nu}_{(i-1)}, T, B), \end{aligned} \tag{62}$$

*where $\mathcal{DC}^{(k_1,k_2,k_3)}(\overline{\nu}, T)$, $\mathcal{DCP}^{(k_1,k_2,k_3)}(\overline{\nu}, T, B)$ and $\mathcal{DICP}^{(k_1,k_2,k_3)}(\overline{\nu}, T, B)$ as in Definitions 4.7, 4.9 and 4.12, respectively.*

Next, we define a generalization of the function $U^k_2$ (see Corollary 4.16).

**Definition 5.3.** *Let $1 \leqslant i \leqslant k$ be a integer. Let $D_0$, $\mathcal{D}_i$ be sets as in Definition 5.1. We define the function $U^i_2 : \mathcal{D}_0 \times \mathcal{D}_1 \times \ldots \times \mathcal{D}_i \to D_i$ given by*

$$U^1_2(\nu_0, \nu_1, T) = (\nu_1 \circ TDTA + \nu_0 \circ TDA - DC)(1 - \overline{\nu}_0 \circ TB)^{-1}, \tag{63}$$

*for* $i \geqslant 2$

$$
\begin{aligned}
(U_2^i)(\overline{\nu}_0, \overline{\nu}_1, \ldots, \overline{\nu}_i) &= (1-\overline{\nu}_0 \circ TB)^{-1} Sym^i \circ \Big(\mathcal{DC}^{(i,i,i)}(\overline{\nu}_i, T)A\Big) \\
&+ (1-\overline{\nu}_0 \circ TB)^{-2} Sym^i \circ \Big(\mathcal{DC}^{(i,1,i-1)}(\overline{\nu}_1, \ldots, \overline{\nu}_{i-1}, T)A - D^i(C)\Big) \\
&+ (1-\overline{\nu}_0 \circ TB)^{-2} Sym^i \circ \Big(\mathcal{DCP}^{(i,0,i-1)}(\overline{\nu}_0, \overline{\nu}_1, \ldots, \overline{\nu}_{(i-1)}, T, A)\Big) \qquad (64)
\end{aligned}
$$

*where* $\mathcal{DC}^{(k_1,k_2,k_3)}(\overline{\nu}, T)$ *and* $\mathcal{DCP}^{(k_1,k_2,k_3)}(\overline{\nu}, T, B)$, *as in Definitions 4.7 and 4.9, respectively.*

Next, we define a generalization of the function $U_3^k$ (see Corollary 4.17).

**Definition 5.4.** *Let* $2 \leqslant i \leqslant k$ *be a integer. Let* $D_0$, $\mathcal{D}_i$ *be sets as in Definition 5.1. We define the function* $U_3^i : \mathcal{D}_0 \times \mathcal{D}_1 \times \ldots \times \mathcal{D}_i \to D_i$ *given by*

$$
(U_3^i)(\overline{\nu}_0, .., \overline{\nu}_i) = \qquad (65)
$$

$$
Sym^i \circ \sum_{q=1}^{i-1} \binom{i}{q} \phi^{(q,i-q)}(\mathcal{DCP}^{(q,0,q)}(\overline{\nu}_0, .., \overline{\nu}_q, T, A) - D^i C, \mathcal{DICP}^{(i-q,1,i-q)}(\overline{\nu}_0, .., \overline{\nu}_{i-q-1}, T, B)),
$$

*where* $\mathcal{DCP}^{(k_1,k_2,k_3)}(\overline{\nu}, T, B)$, $\mathcal{DICP}^{(k_1,k_2,k_3)}(\overline{\nu}, T, B))$, $\phi^{(q,i-q)}$ *as in Definitions 4.9, 4.12 and 4.6, respectively.*

Next, we define a generalization of the function $D^k\Gamma(\nu)$ (see Lemma 4.18).

**Definition 5.5.** *Let* $1 \leqslant i \leqslant k$ *be a integer. Let* $D_0$, $\mathcal{D}_i$ *be sets as in Definition 5.1, and assume the functions* $U_1^i, U_2^i$ *and* $U_3^i$ *as in Definitions 5.2, 5.3 and 5.4, respectively. We define the function* $\Psi^i : \mathcal{D}_0 \times \mathcal{D}_1 \times \ldots \times \mathcal{D}_i \to D_i$ *given by*

$$
\Psi^1(\overline{\nu}_0, \overline{v}_1, \overline{\nu}_2) = (U_1^1 + U_2^1)(\overline{\nu}_0, \overline{\nu}_1), \qquad (66)
$$

*and for* $i \geqslant 2$

$$
\Psi^i(\overline{\nu}_0, \overline{\nu}_1, \ldots, \overline{\nu}_i) = (U_1^i + U_2^i + U_3^i)(\overline{\nu}_0, \overline{\nu}_1, \ldots, \overline{\nu}_i). \qquad (67)
$$

**Remark 5.6.** *The functions* $\Psi^i$ *for the cases* $i = 1$ *and* $i = 2$ *were established in* [SS94, Eq. (13)] *and* [MPP00, Eq. (42)] *respectively.*

**Proposition 5.7.** *Let* $1 \leqslant i \leqslant k$ *be a integer. Then, the function* $\Psi^i$ *given in Definition 5.5 is well-defined. Moreover, if* $\overline{\nu}_0 \in \mathcal{A}_L$ *is of class* $C^i$, *then*

$$
\Psi^i(\overline{\nu}_0, D\overline{\nu}_0, \ldots, D^i\overline{\nu}_0) = D^i\Gamma(\overline{\nu}_0). \qquad (68)
$$

*Proof.* To prove that the function $\Psi^i$ is well-defined, it suffices to show that

$$
\Psi^i(\overline{\nu}_0, \ldots, \overline{\nu}_i) \in \mathcal{D}_i, \quad \text{for all} \quad \overline{\nu}_j \in \mathcal{D}_j, 0 \leqslant j \leqslant 1. \qquad (69)
$$

That is, by Definition 5.1 we must to show that

(a) $\Psi^i(\overline{\nu}_0, \ldots, \overline{\nu}_i)$ is continuous on $D$ and
(b) $\Psi^i(\overline{\nu}_0, \ldots, \overline{\nu}_i)(x, 0) = 0$, for every $x \in \mathbb{R}^n$, $\|x\| \leqslant 1$,

for all $\overline{\nu}_j \in \mathcal{D}_j, 0 \leqslant j \leqslant 1$. Indeed, by Definition 5.5 we have that $\Psi^i(\overline{\nu}_0, \ldots, \overline{\nu}_i)$ is continuous on $D^*$, so it remains to show the continuity of $\Psi^i(\overline{\nu}_0, \ldots, \overline{\nu}_i)$ at the points $(x, 0) \in D_0$. Analysis similar to that in the proof of Proposition 3.7 shows that

$$\lim_{(a,b)\to(x,0)} \psi^i(\overline{\nu}_0, \ldots, \overline{\nu}_i)(a, b) = 0, \tag{70}$$

for all $(x, 0) \in D_0$. Therefore, if we define

$$W^i(x, y) = \begin{cases} \Psi^i(\overline{\nu}_0, \ldots, \overline{\nu}_i)(x, y), & y \neq 0, \\ 0, & y = 0, \end{cases}$$

then, we get a continuous extension of $\Psi^i(\overline{\nu}_0, \ldots, \overline{\nu}_i)$ on $D$, which completes the proof of (5.1), so $\Psi^i(\overline{\nu}_0, \ldots, \overline{\nu}_i) \in \mathcal{D}_i$. Therefore $\Psi^i$ is well-defined. The equality in Eq. (68) follows from Definition 5.5 and Lemma 4.18. This concludes the proof. ■

5.2. **Part 2: The function $\widetilde{N}_i$.** In this sub-section will be shown the following proposition.

**Proposition 5.8.** *Let $1 \leqslant i \leqslant k$ be a integer. Let $\Psi^j, 1 \leqslant j \leqslant i$ be functions as in Definition 5.5. Then the function*

$$\widetilde{N}_i : \mathcal{D}_0 \times \mathcal{D}_1 \times \ldots \times \mathcal{D}_i \to \mathcal{D}_0 \times \mathcal{D}_1 \times \ldots \times \mathcal{D}_i \tag{71}$$

*given by*

$$\widetilde{N}_i(\overline{\nu}_0, \overline{\nu}_1, \ldots, \overline{\nu}_i) = (\Gamma(\overline{\nu}_0), \Psi^1(\overline{\nu}_0, \overline{\nu}_1), \ldots, \Psi^i(\overline{\nu}_0, \overline{\nu}_1, \ldots, \overline{\nu}_i)), \tag{72}$$

*have a global attracting fixed point $(A_0, A_1, \ldots, A_i)$.*

5.2.1. *Preliminares.* Before proving Proposition 5.8, we state without proof two theorem which will be useful in the sequel.

**Theorem 5.9** (Fiber Contraction Theorem [HP69])**.** *Let $(X, d_X)$ and $(Y, d_Y)$ be two complete metric spaces, and let $\Upsilon : X \times Y \to X \times Y$ be a map of the form*

$$\Upsilon(x, y) = (\Gamma(x), \Psi(x, y)).$$

*Assume that*

(a) *$\Gamma$ has an attracting fixed point $x_\infty$, that is,*
$$\Gamma(x_\infty) = x_\infty, \lim_{n\to\infty} \Gamma^n(x) = x_\infty, \quad \textit{for all} \quad x \in X;$$
(b) *the family of functions $\Psi^y : X \to Y$ given by $\Psi^y(x) = \Psi(x, y)$ depends on $y$ continuously; that is, if $x_n \to x$ as $n \to \infty$, then $\Psi^y(x_n) \to \Psi^y(x)$ as $n \to \infty$.*
(c) *for every $x \in X$ the map $\Psi_x := \Psi(x, .) : Y \to Y$ defined by $\Psi_x(y) := \Psi(x, y)$ is a $\lambda$-contraction, with $\lambda < 1$. This mean that*
$$d_Y(\Psi_x(y_1), \Psi_x(y_2)) \leqslant \lambda d_Y(y_1, y_2),$$
*for all $x \in X$ and $y_1, y_2 \in Y$.*

*Then, if $y_\infty$ denotes the unique fixed point of $\Psi_{x_\infty}$, the point $(x_\infty, y_\infty) \in X \times Y$ is an attracting fixed point of $\Upsilon$, that is,*

$$\lim_{n\to\infty} \Upsilon^n(x,y) = (x_\infty, y_\infty).$$

**Theorem 5.10** (Perron-Frobenius Theorem for positive matrices [Mey00])**.** *Let $A = [a_{i,j}]_{n\times n}$ be a real $n \times n$ positive matrix: $a_{i,j} > 0$, for $1 \leqslant i, j \leqslant n$. Then:*

(a) *$A$ has a positive simple eigenvalue $r$ which is equal to the spectral radius of $A$.*
(b) *There exists an eigenvector $x$ with all the coordinates positives such that $Ax = rx$.*
(c) *The eigenvector is the unique vector defined by*
$$Ap = rp,\ p > 0,\ \text{and}\ \| p \|_1 = 1,\ \text{where}\ \| p \|_1 = \textstyle\sum_{i=1}^n |p_i|,$$
*and, except for positive multiples of $p$, there are no other nonnegative eigenvector for $A$, regardless of the eigenvalue.*
(d) *An estimate of $r$ is given by inequalities:*
$$\min_i \sum_j a_{ij} \leqslant r \leqslant \max_i \sum_j a_{ij}.$$

We will now given some elementary properties of multilinear maps. Let us start by fixing the notion. The set $\{1, ..., n\}$ will be denoted by $[n]$. If $E := \mathbb{R}^n$ and $F := \mathbb{R}$, then $\mathcal{F}([k], \{E, F\})$ denoted the set of all the functions $f : [k] \to \{E, F\}$. Notice that the cardinality of $\mathcal{F}([k], \{E, F\})$ is $2^k$. Finally $\pi_E : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ and $\pi_F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ denoted the projections of $\mathbb{R}^{n+1}$ on $E$ along $F$ and of $\mathbb{R}^{n+1}$ on $F$ along $E$ respectively.

**Definition 5.11.** *Assume that $f \in \mathcal{F}([k], \{E, F\})$ and that $\heartsuit = E$ or $\heartsuit = F$. Then, define $g_{f,\heartsuit} : [n+1] \to \{E, F\}$ by*

$$g_{f,\heartsuit}(i) = \begin{cases} f(i), & \text{if} \quad i \in [n], \\ \heartsuit, & \text{if} \quad i = n+1. \end{cases}$$

*By $\Omega([n+1], \{\heartsuit\})$ we denote the set of all functions $g_{f,\heartsuit}$.*

By $A \uplus B$ we denote the usual disjoint intersection between sets.

**Lemma 5.12.** *The following statement holds:*

(a) $\Omega([n+1], \{E\}) \uplus \Omega([n+1], \{F\}) = \mathcal{F}([n+1], \{E, F\})$.

*Proof.* The proof follows immediately from Definition 5.11. ∎

Recall that $L^k(\mathbb{R}^{n+1}, \mathbb{R}^n)$ denoted the space of all the $k$-linear maps from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$.

**Definition 5.13.** *Assume that $f \in \mathcal{F}([k], \{E, F\})$. Then, the set of all $k$-linear maps $b$ such that*

(a) *$b(\pi_{g(1)}(x_1), \pi_{g(2)}(x_2), \dots \pi_{g(k)}(x_k)) = 0$, for every $g \in \mathcal{F}([k], \{E, F\})$, $g \neq f$ and for each $k$-tuple $(x_1 \dots, x_k) \in \underbrace{\mathbb{R}^{n+1} \times \dots \times \mathbb{R}^{n+1}}_{k-times}$.*

*will be denoted by*

$$L_f^k(\mathbb{R}^{n+1}, \mathbb{R}^n)$$

**Lemma 5.14.** *We have the following properties:*

(a) *If* $b \in L^k(\mathbb{R}^{n+1}, \mathbb{R}^n)$, *then*

$$b(x_1, x_2, \ldots, x_k) = \sum_{f \in \mathcal{F}([k],\{E,F\})} b(\pi_{f(1)}(x_1), \ldots, \pi_{f(k)}(x_k)),$$

*for every* $(x_1, \ldots, x_k) \in \underbrace{\mathbb{R}^{n+1} \times \ldots \times \mathbb{R}^{n+1}}_{k-times}$.

*The function* $(x_1, x_2, \ldots, x_k) \to b(\pi_{f(1)}(x_1), \ldots, \pi_{f(k)}(x_k))$ *will be denoted by* $b_f$.

(b) *If* $f \in \mathcal{F}([k], \{E, F\})$ *and* $b \in L_f^k(\mathbb{R}^{n+1}, \mathbb{R}^n)$, *then*

$$b(x_1, x_2, \ldots, x_k) = b(\pi_{f(1)}(x_1), \pi_{f(2)}(x_2), \ldots, \pi_{f(k)}(x_k)),$$

*for every* $(x_1, \ldots, x_k) \in \underbrace{\mathbb{R}^{n+1} \times \ldots \times \mathbb{R}^{n+1}}_{k-times}$.

(c) $L^k(\mathbb{R}^{n+1}, \mathbb{R}^n)$ *can be decomposed into a direct sum of* $2^k$ *k-linear maps that is,*

$$L^k(\mathbb{R}^{n+1}, \mathbb{R}^n) = \bigoplus_{f \in \mathcal{F}([k],\{E,F\})} L_f^k(\mathbb{R}^{n+1}, \mathbb{R}^n),$$

*where* $L_f^k(\mathbb{R}^{n+1}, \mathbb{R}^n)$ *as in Definition 5.13.*

*Proof.* The proof follows from Lemma 5.12 and Eq. (5.13). ■

5.2.2. *Proof of Proposition 5.8.* In order to prove Proposition 5.8 we state and prove the following proposition.

**Proposition 5.15.** *Under the notation of Definitions 5.1 and* 5.5. *Let* $1 \leqslant i \leqslant k$ *be a integer and fix a point* $(\overline{\nu}_0, \ldots, \overline{\nu}_{i-1}) \in \mathcal{D}_0 \times \mathcal{D}_1 \times \ldots \times \mathcal{D}_{i-1}$. *Then, the space* $D_i$ *can be endowed with a norm* $|.|_{i,D}$ *equivalent to the original norm* $||.||_D$ *such that the function*

$$\Psi^i(\overline{\nu}_0, \ldots, \overline{\nu}_{i-1}, \bullet) : \mathcal{D}_i \to \mathcal{D}_i$$

*is a contraction with constant of contraction independent of the point* $(\overline{\nu}_0, \overline{\nu}_1, \ldots, \overline{\nu}_{i-1})$.

The proof of Proposition 5.15 will be given after some lemmas. We set,

$$\widehat{DT}(x, y) := \begin{bmatrix} A(x, y) & B(x, y) \\ C(x, y) & 1 \end{bmatrix}_{(n+1)\times(n+1)}, \tag{73}$$

where the functions $A(x, y), B(x, y)$ and $C(x, y)$ are as in Definition 2.1.

**Lemma 5.16.** *Let* $M^i : L^i(\mathbb{R}^{n+1}, \mathbb{R}^n) \to L^i(\mathbb{R}^{n+1}, \mathbb{R}^n)$ *be a map defined by*

$$M^i(b)(x_1, \ldots, x_i) = b(\widehat{DT}x_1, \ldots, \widehat{DT}x_i).$$

*Then, the space* $L^i(\mathbb{R}^{n+1}, \mathbb{R}^n)$ *can be endowed with a norm* $|.|_i$ *equivalent to* $\|.\|$ *such that*

$$\frac{|M^i(b)|_i}{|b|_i} \leqslant \max_{\substack{m,n\in\mathbb{N} \\ m+n=i}} \{(||A||_D + ||B||_D)^m(||C||_D + 1)^n\}. \tag{74}$$

*Proof.* Through of the proof, we deal with the case that $\|B\|_D$ is nonzero, the other case is similar. We will endow $L^i(\mathbb{R}^{n+1}, \mathbb{R}^n)$ with a new norm $|.|_i$ in the following way: letting

$$c_{g,f} := ||\pi_{g(1)}\widehat{DT}\pi_{f(1)}|| \ldots ||\pi_{g(i)}\widehat{DT}\pi_{f(i)}||, \tag{75}$$

where $g$ and $f \in \mathcal{F}([i], \{E, F\})$ while

$$\pi_{g(j)}\widehat{DT}\pi_{f(j)} := \begin{cases} A, & \text{if } g(j) = E \text{ and } f(j) = E, \\ B, & \text{if } g(j) = E \text{ and } f(j) = F, \\ C, & \text{if } g(j) = F \text{ and } f(j) = E, \\ 1, & \text{if } g(j) = F \text{ and } f(j) = F. \end{cases} \tag{76}$$

Next up, consider the matrix

$$\Delta := [c_{g,f}]_{2^i\times 2^i}. \tag{77}$$

Notice that since, by assumption $||A||_D, ||B||_D$ and $||C||_D$ are nonzero, then, in view of (75) and (76) it follows that $c_{g,f} > 0$, where $g, f \in \mathcal{F}([i], \{E, F\})$. Thus, the matrix $\Delta$ is positive. Therefore, by Perron-Frobenius Theorem 5.10 applied to matrix $\Delta$, we get

(a) The matrix $\Delta$ has a positive eigenvalue $\lambda$.
(b) The matrix $\Delta$ has an eigenvector $V$ with entries $k_f$ such that

$$\sum_{f\in\mathcal{F}([i],\{E,F\})} k_f = 1.$$

(c) An estimate of $\lambda$ is given by inequalities

$$\min_g \sum_f c_{g,f} \leqslant \lambda \leqslant \max_g \sum_f c_{g,f}. \tag{78}$$

Let $b \in L^i(\mathbb{R}^{n+1}, \mathbb{R}^n), b \neq 0$. In view of Lemma 5.14(a) we can write

$$b = \sum_{f\in\mathcal{F}([i],\{E,F\})} b_f, \tag{79}$$

where $b_f$ is as in as in Definition 5.13. Thus, we can define the norm $|.|_i$ on $L^i(\mathbb{R}^{n+1}, \mathbb{R}^n)$ by

$$|b|_i := \sum_{f\in\mathcal{F}([i],\{E,F\})} k_f\|b_f\|. \tag{80}$$

It is easily to seen that $|.|_i$ is a norm on $L^i(\mathbb{R}^{n+1}, \mathbb{R}^n)$ equivalent to the norm $||.||$.
We now will prove that

$$\frac{|M^i(b)|_i}{|b|_i} \leqslant \max_{\substack{m,n\in\mathbb{N} \\ m+n=i}} \{(||A||_D + ||B||_D)^m(||C||_D + 1)^n\}. \tag{81}$$

Indeed, by definition one has $M^i(b)$ is $i$-linear map, then on account of Lemma 5.14(c) and (80) we have

$$|M^i(b)|_i := \sum_{f\in\mathcal{F}([i],\{E,F\})} k_f \|M^i(b)_f\|, \tag{82}$$

where $M^i(b)_f$ as in Definition 5.13. But, by using Lemma 5.14(b) we have

$$M^i(b)_f(x_1,\ldots,x_i) = M^i(b)(\pi_{f(1)}(x_1),\ldots,\pi_{f(i)}(x_i)) \tag{83}$$

and by assumption (5.16) we have

$$M^i(b)(x_1,\ldots,x_i) = b(\widehat{DT}x_1,\ldots,\widehat{DT}x_i). \tag{84}$$

Thus, combining (84) and (83) we get

$$M^i(b)_f(x_1,\ldots,x_i) = b(\widehat{DT}\pi_{f(1)}x_1,\ldots,\widehat{DT}\pi_{f(i)}x_i). \tag{85}$$

Furthermore, by using Lemma 5.14(a) we can write

$$b(\widehat{DT}\pi_{f(1)},\ldots,\widehat{DT}\pi_{f(i)}) = \sum_{g\in\mathcal{F}([i],\{E,F\})} b_g(\pi_{g(1)}\widehat{DT}\pi_{f(1)},\ldots,\pi_{g(i)}\widehat{DT}\pi_{f(i)}). \tag{86}$$

Therefore, it follows from (86) and (85), that

$$M^i(b)_f(x_1,\ldots,x_i) = \sum_{g\in\mathcal{F}([i],\{E,F\})} b_g(\pi_{g(1)}\widehat{DT}\pi_{f(1)}(x_1),\ldots,\pi_{g(i)}\widehat{DT}\pi_{f(i)}(x_i)). \tag{87}$$

Hence, on account of (82) we get

$$|M^i(b)|_i \leqslant \sum_{f\in\mathcal{F}([i],\{E,F\})} k_f \sum_{g\in\mathcal{F}([i],\{E,F\})} \|b_g(\pi_{g(1)}\widehat{DT}\pi_{f(1)},\ldots,\pi_{g(i)}\widehat{DT}\pi_{f(i)})\|. \tag{88}$$

Since $b_g$ is $i$-linear map, we have

$$||b_g(\pi_{g(1)}\widehat{DT}\pi_{f(1)},\ldots,\pi_{g(i)}\widehat{DT}\pi_{f(i)})|| \leqslant \underbrace{||\pi_{g(1)}\widehat{DT}\pi_{f(1)}||\ldots||\pi_{g(i)}\widehat{DT}\pi_{f(i)}||}_{:=c_{g,f}}.||b_g||.$$

Consequently, Eq. (88) becomes

$$|M^i(b)|_i \leqslant \sum_{g\in\mathcal{F}([i],\{E,F\})} ||b_g|| \sum_{f\in\mathcal{F}([i],\{E,F\})} c_{g,f}k_f. \tag{89}$$

Notice that, since $V = [k_f]_{f\in\mathcal{F}([i],\{E,F\})}$ is an eigenvector of matrix $\Delta$, we have

$$\Delta[k_f]_{f\in\mathcal{F}([i],\{E,F\})} = \lambda[k_f]_{f\in\mathcal{F}([i],\{E,F\})}, \tag{90}$$

where $\Delta = [c_{g,f}]$ while $g$ and $f \in \mathcal{F}([i],\{E,F\})$.
Hence, if we fix $g \in \mathcal{F}([i],\{E,F\})$ it is easily seen that

$$\sum_{f\in\mathcal{F}([i],\{E,F\})} c_{g,f}k_f = \lambda k_g. \tag{91}$$

Thus, by replacing (91) into (89) we get

$$|M^i(b)|_i \leqslant \lambda \sum_{g\in\mathcal{F}([i],\{E,F\})} ||b_g||k_g. \tag{92}$$

Moreover, by definition we can write

$$|b|_i = \sum_{g\in\mathcal{F}([i],\{E,F\})} ||b_g||k_g. \tag{93}$$

Therefore, from (93) and (92) one obtains

$$|M^i(b)|_i \leqslant \lambda|b|_i. \tag{94}$$

Through of the remainder of the proof, we denote by $\#(S)$ the cardinality of the set $S$.

**Claim 5.17.** *Let $f$ and $g \in \mathcal{F}([i],\{E,F\})$ such that $\#(g^{-1}(E)) = m$ and $\#(g^{-1}(F)) = n$. Then the following equality holds:*

$$\sum_{f\in\mathcal{F}([i],\{E,F\}} c_{g,f} = (||A||_D + ||B||_D)^m(||C||_D + 1)^n, \tag{95}$$

*where $c_{g,f}$ as in Eq.* (75)*.*

Proof of the Claim. Since $\#(g^{-1}(E)) = m$ and $\#(g^{-1}(F)) = n$, then one can consider $g^{-1}(E) := \{a_1, a_2, \ldots, a_m\}$ and $g^{-1}(F) := \{b_1, b_2, \ldots, b_n\}$. Thus, by definition we have

$$\pi_{g(a_i)}\widehat{DT}\pi_{f(a_i)} := \begin{cases} A, & \text{if } f(a_i) = E, \\ B, & \text{if } f(a_i) = F, \end{cases} \tag{96}$$

and

$$\pi_{g(b_i)}\widehat{DT}\pi_{f(b_i)} := \begin{cases} C, & \text{if } f(b_i) = E, \\ 1, & \text{if } f(b_i) = F. \end{cases} \tag{97}$$

Now, consider integers $s, t$ with $0 \leqslant s \leqslant m$, $0 \leqslant t \leqslant n$ and take $f \in \mathcal{F}([i],\{E,F\})$ such that

$$\#(g^-(E) \cap f^{-1}(E)) = s \quad \text{and} \quad \#(g^-(F) \cap f^{-1}(E)) = t.$$

Then, since $c_{g,f} := ||\pi_{g(1)}\widehat{DT}\pi_{f(1)}|| \ldots ||\pi_{g(i)}\widehat{DT}\pi_{f(i)}||$, it follows from (97) and (96) that

$$c_{g,f} = ||A||^s_D||B||^{m-s}_D||C||^t_D 1^{n-t}. \tag{98}$$

In addition, since $\#g^{-1}(E) = m$ and $\#g^{-1}(F) = n$, it is not difficult to see that the cardinality of the sets

$$\mathcal{F}_{g,s}([i],\{E,F\}) := \{f \in \mathcal{F}([i],\{E,F\}) : card(g^-(E) \cap f^{-1}(E)) = s\} \tag{99}$$

and

$$\mathcal{F}_{g,t}([i],\{E,F\}) := \{f \in \mathcal{F}([i],\{E,F\}) : card(g^-(E) \cap f^{-1}(E)) = t\} \tag{100}$$

are $\binom{n}{t}$ and $\binom{m}{s}$, respectively. Thus, from (100) and (99), on account of Rule of Product [Coh78, p. 13] we deduce that the cardinality of

$$\mathcal{F}_{g,st}([i],\{E,F\}) := \mathcal{F}_{g,s}([i],\{E,F\}) \cap \mathcal{F}_{g,t}([i],\{E,F\}) \tag{101}$$

is $\binom{m}{s}.\binom{n}{t}$. Finally, notice that

$$\mathcal{F}([i],\{E,F\}) = \biguplus_{\substack{0\leqslant s\leqslant m\\ 0\leqslant t\leqslant n}} \mathcal{F}_{g,st}([i],\{E,F\}).$$

Whence, on account of (98) and Binomial Theorem we get the following chain of equalities

$$
\begin{aligned}
\sum_{f\in\mathcal{F}([i],\{E,F\})} c_{g,f} &= \sum_{s=0}^{m}\sum_{t=0}^{n}\left(\sum_{f\in\mathcal{F}_{g,st}([i],\{E,F\})} c_{g,f}\right) \\
&= \sum_{s=0}^{m}\sum_{t=0}^{n}\binom{m}{s}.\binom{n}{t}||A||_D^s||B||_D^{m-s}||C||_D^t 1^{n-t} \\
&= (||A||_D+||B||_D)^m(||C||_D+1)^n.
\end{aligned}
$$

Thus Claim 5.17 is proved.

Finally, from (78) and Claim 5.17 we conclude that

$$
\frac{|M^i(b)|_i}{|b|_i} \leqslant \max_{\substack{m,n\in\mathbb{N}\\ m+n=i}} \{(||A||_D+||B||_D)^m(||C||_D+1)^n\}, \tag{102}
$$

for all $b\in L^i(\mathbb{R}^{n+1},\mathbb{R}^n)$, which concludes the proof of the lemma. ■

Now, we are going to prove Proposition 5.15 mentioned in the beginning of the sub-section, which we recall here. Before that is important recall that

$$
\mathcal{D}_i := \{\nu_i : D\to L_s^i(\mathbb{R}^{n+1},\mathbb{R}^n) : \nu_i(x,0)=0; v_i \text{ is continuous}\}.
$$

**Proposition 5.18.** *Under the notation of Definitions 5.1 and* 5.5. *Let* $1\leqslant i\leqslant k$ *be a integer and fix a point* $(\overline{\nu}_0,\ldots,\overline{\nu}_{i-1})\in\mathcal{D}_0\times\mathcal{D}_1\times\ldots\times\mathcal{D}_{i-1}$. *Then, the space* $D_i$ *can be endowed with a norm* $|.|_{i,D}$ *equivalent to the original norm* $||.||_D$ *so that the function*

$$
\Psi^i(\overline{\nu}_0,\ldots,\overline{\nu}_{i-1},\bullet):\mathcal{D}_i\to\mathcal{D}_i
$$

*it is a contraction with constant of contraction independent of the point* $(\overline{\nu}_0,\overline{\nu}_1,\ldots,\overline{\nu}_{i-1})$.

***Proof of Proposition 5.18***. Let $\nu_i\in\mathcal{D}_i$. We define its norm to be

$$
|\nu_i|_{i,D} := \sup\{|\nu_i(x,y)|_i : (x,y)\in D\}, \tag{103}
$$

where $|.|_i$ is the norm $|.|_i$ on $L^i(\mathbb{R}^{n+1},\mathbb{R}^n)$ as in Lemma 5.16. It is easy to check that $|.|_{i,D}$ is a norm on $\mathcal{D}_i$ equivalent to $\|.\|_D$.

Let $\nu_i^1=\Psi^i(\overline{\nu}_0,\overline{\nu}_1,\ldots,\overline{\nu}_{i-1},\mu^1)$ and $\nu_i^2=\Psi^i(\overline{\nu}_0,\overline{\nu}_1,\ldots,\overline{\nu}_{i-1},\mu^2)$, where $\mu^1,\mu^2\in\mathcal{D}_i$. From Definition 5.5 one can deduce that

$$
\begin{aligned}
\nu_i^1-\nu_i^2 &= (\overline{\nu}_0\circ TA-C)i!(1-\overline{\nu}_0\circ TB)^{-2}.\mathcal{DC}^{(i,i,i)}((\mu^1-\mu^2),T)B \\
&+ (1-\overline{\nu}_0\circ B)^{-1}\mathcal{DC}^{(i,i,i)}((\mu^1-\mu^2),T)A.
\end{aligned} \tag{104}
$$

Recall that, by Eq. (14) we have

$$
\mathcal{DC}^{(i,i,i)}((\mu^1-\mu^2),T)(x,y) := i!\partial_y G(x,y)(\mu^1-\mu^2)\circ T(x,y)\underbrace{\widehat{DT}(x,y)\ldots\widehat{DT}(x,y)}_{k-times}, \tag{105}
$$

where $\widehat{DT}(x,y)$ is as in Eq. (73). Hence, in view of (104), (103) and Lemma 5.16 we get

$$\begin{aligned}
\left|\nu_i^1-\nu_i^2\right|_{i,D} &\leqslant \left|(\mu_i^1-\mu_i^2)\right|_{i,D}\frac{(L||A||_D+||C||_D)||B||_D}{\|\partial_y G(x,y)\|^{-i}(1-L||B||_D)^2}(i!)^2\Lambda(i)\\
&+\left|(\mu^1-\mu^2)\right|_{i,D}\frac{||A||_D(1-L||B||_D)}{||\partial_y G(x,y)||^{-i}(1-L||B||_D)^2}(i!)^2\Lambda(i)\\
&= (i!)^2\left|(\mu^1-\mu^2)\right|_{i,D}\frac{||A||_D+||C||_D||B||_D}{||\partial_y G(x,y)||^{-i}(1-L||B||_D)^2}\Lambda(i), \qquad (106)
\end{aligned}$$

where

$$\Lambda(i) := \max_{\substack{m,n\in\mathbb{N}\\ m+n=i}}\{(||A||_D+||B||_D)^m(||C||_D+1)^n\}.$$

But, from Eq. (5) we have $2||B||_D L := 1-||A||_D-\sqrt{(1-||A||_D)^2-4||B||_D||C||_D}$. Hence, Eq. (106) becomes

$$\left|\nu_i^1-\nu_i^2\right|_{i,D} \leqslant \left|(\mu^1-\mu^2)\right|_{i,D}\Theta(i), \qquad (107)$$

where

$$\Theta(i) := \frac{(||A||_D+||C||_D||B||_D)\max\limits_{m+n=i}\{(||A||_D+||B||_D)^m(||C||_D+1)^n\}}{(2i!)^{-2}||\partial_y G||^{-i}\left(1+||A||_D+\sqrt{(1-||A||_D)^2-4||B||_D||C||_D}\right)^2}.$$

Moreover, by using Assumption 2.2($L_3$) one can see that

$$\Theta(i)<1, \quad 1\leqslant i\leqslant k. \qquad (108)$$

Therefore, on account of Eq. (107) one obtains that the function

$$\Psi^i(\overline{\nu}_0,\overline{\nu}_1,\ldots,\overline{\nu}_{i-1},\bullet):\mathcal{D}_i\to\mathcal{D}_i$$

is a contraction independent of the point $(\overline{\nu}_0,\overline{\nu}_1,\ldots,\overline{\nu}_{i-1})$, which finishes the proof. ∎

Before proceeding to state and prove the following lemma, it is convenient to introduce some useful notation. Consider the following norm-spaces $X_1,\ldots,X_n$ with norm $\|.\|_i$, for $0\leqslant i\leqslant k$ respectively and let $X := X_1\times\ldots\times X_n$. Then the norm of the space $X$ will be denoted by $\|.\|_X$ and defined by $\|.\|_X := \max\{\|.\|_i : 1\leqslant i\leqslant n\}$.

**Lemma 5.19.** *Under Definition 5.5. Let $0\leqslant i\leqslant k$ be a integer. Suppose that the sets $\mathcal{D}_j$, for $1\leqslant j\leqslant i$ are endowed with the norm $|.|_{j,D}$ from Proposition 5.15 and the set $X_i := \mathcal{D}_0\times\mathcal{D}_1\times\ldots\times\mathcal{D}_i$ is endowed with the norm $|.|_{X_i}$. Then, the family of maps $\Psi^{(i,\overline{\nu}_i)}: X_{i-1}\to\mathcal{D}_i$ given by $\Psi^{(i,\overline{\nu}_i)}(\nu_0,\nu_1,\ldots,\nu_{i-1}) = \Psi^i(\nu_0,\nu_1,\ldots,\nu_{i-1},\overline{\nu}_i)$ depends on $\overline{\nu}_i$ continuously in the following sense: if $(\nu_0^n,\nu_1^n,\ldots,\nu_{i-1}^n)\to(\nu_0,\nu_1,\ldots,\nu_{i-1})$ as $n\to\infty$ in the space $X_{i-1}$, then $\Psi^i(\nu_0^n,\nu_1^n,\ldots,\nu_{i-1}^n,\overline{\nu}_i)\to\Psi^i(\nu_0,\nu_1,\ldots,\nu_{i-1},\overline{\nu}_i)$ in the space $\mathcal{D}_i$ for any fixed $\overline{\nu}_i\in\mathcal{D}_i$.*

*Proof.* The proof follows from Definitions 5.5, 5.4, 5.3 and 5.2. ∎

We are going to prove Proposition 5.15, which we recall here.

**Proposition 5.20.** *Assume the notation of Lemma 5.19. Then, the function*

$$\widetilde{N}_i : X_i \to X_i$$

*defined by*

$$\widetilde{N}_i(\overline{\nu}_0, \overline{v}_1, \ldots, \overline{\nu}_i) = (\Gamma(\overline{\nu}_0), \Psi^1(\overline{\nu}_0, \overline{\nu}_1), \ldots, \Psi^i(\overline{\nu}_0, \overline{\nu}_1, \ldots, \overline{\nu}_i))$$

*has a global attracting fixed point* $(A_0, A_1, \ldots, A_i)$.

*Proof.* We proceed by induction on $i$. Suppose that the statement holds for $j$ with $0 \leqslant j < i$. We wish to show the statement holds for $i$. To do this; will be proved that the map $\widetilde{N}_i = (\widetilde{N}_{i-1}, \Psi^i) : X_{i-1} \times Y \to X \times Y$, where $X_{i-1} = \mathcal{D}_0 \times \mathcal{D}_1 \times \ldots \times \mathcal{D}_{i-1}$ and $Y = \mathcal{D}_i$, satisfies the three conditions conditions of Fiber Contraction Theorem 5.9. Indeed,

(a) By inductive hypothesis the function $\widetilde{N}_{i-1} : X_{i-1} \to X_{i-1}$ has a global attracting fixed point $(A_0, \ldots, A_{i-1}) \in X_{i-1}$.
(b) By using Theorem 5.15 applied to $(A_0, \ldots, A_{i-1})$, we have that

$$\Psi^i(A_0, \ldots, A_{i-1}, \bullet) : \mathcal{D}_i \to \mathcal{D}_i$$

is a contraction. Then by the Banach fixed-point theorem $\Psi^i(A_0, \ldots, A_{i-1}, \bullet)$ has an attracting fixed point $A_i$.
(c) It follows from Lemma 5.19 that $\Psi^i(., A_i) : X \to Y$ is continuous.

Therefore, from $(a), (b)$, and $(c)$, we deduce that $\widetilde{N}_i : X_i \times X_i$ satisfies the three conditions of Theorem 5.9. Thus, we conclude that there exists a global attracting fixed point $(A_0, A_1, \ldots, A_i)$ to the function $\widetilde{N}_i$, which completes the proof. ∎

Now we are ready to prove the Proposition 3.8, which we recall here.

**Proposition 5.21.** *If* $\overline{\nu} \in \mathcal{A}_L$ *is a* $C^k$ *function and* $D^i\overline{\nu}(x, 0) = 0, 0 \leqslant i \leqslant k$ *and* $(x, 0) \in D_0$. *Then the following limit exists*

$$\lim_{n\to\infty} (\Gamma^n(\overline{\nu}), D(\Gamma^n(\overline{\nu})), \ldots, D^k(\Gamma^n(\overline{\nu}))) = (\nu^*, A_1, A_2, \ldots, A_k),$$

*where* $A_1, A_2, \ldots, A_k$ *are continuous functions.*

***Proof of Theorem 3.8***. Let $\overline{\nu} \in \mathcal{A}_L$ be a $C^k$ function such that $D^i\overline{\nu}(x, 0) = 0$, for all $0 \leqslant i \leqslant k$ and $(x, 0) \in D_0$. By induction, it follows that

$$\widetilde{N}_i^n(\overline{\nu}, D\overline{\nu}, \ldots, D^i\overline{\nu}) = (\Gamma^n(\overline{\nu}), D(\Gamma^n(\overline{\nu})), \ldots, D^i(\Gamma^n(\overline{\nu}))).$$

Hence, on account of Proposition 5.15 one obtains

$$\lim_{n\to\infty} (\Gamma^n(\overline{\nu}), D(\Gamma^n(\overline{\nu})), \ldots, D^k(\Gamma^n(\overline{\nu}))) = (\nu^*, A_1, A_2, \ldots, A_k),$$

where $A_j \in \mathcal{D}_j$, for all $1 \leqslant j \leqslant k$, which concludes the proof. ∎

## References


[ABS77] V. S Afraĭmovič, V. V. Bykov, and L. P. Sil′nikov. The origin and structure of the Lorenz attractor. *Dokl. Akad. Nauk SSSR*, 234(2):336–339, 1977.

[AP87] V. S. Afraĭmovich and Ya. B. Pesin. Dimension of Lorenz type attractors. In *Mathematical physics reviews, Vol. 6*, volume 6 of *Soviet Sci. Rev. Sect. C Math. Phys. Rev.*, pages 169–241. Harwood Academic Publ., Chur, 1987.

[AP10] Vítor Araújo and Maria José Pacifico. *Three-dimensional flows*, volume 53 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, Heidelberg, 2010. With a foreword by Marcelo Viana.

[AV12] Vítor Araújo and Paulo Varandas. Robust exponential decay of correlations for singular-flows. *Comm. Math. Phys.*, 311(1):215–246, 2012.

[Coh78] Daniel I. A. Cohen. *Basic techniques of combinatorial theory*. John Wiley & Sons, New York-Chichester-Brisbane, 1978.

[Die69] J. Dieudonné. *Foundations of modern analysis*. Academic Press, New York-London, 1969. Enlarged and corrected printing, Pure and Applied Mathematics, Vol. 10-I.

[GH83] John Guckenheimer and Philip Holmes. *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, volume 42 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[Guc76] J. Guckenheimer. A strange, strange attractor, in the Hopf bifurcation and its applications. 19:368–381, 1976.

[GW79] John Guckenheimer and R. F. Williams. Structural stability of Lorenz attractors. *Inst. Hautes Études Sci. Publ. Math.*, (50):59–72, 1979.

[HP69] Morris W. Hirsch and Charles C. Pugh. Stable manifolds for hyperbolic sets. *Bull. Amer. Math. Soc.*, 75:149–152, 1969.

[Jak81] M. Jakobson. Absolutely continuous invariant measures for one-parameter families of one-dimensional maps. *Comm. Math. Phys.*, 81:39–88, 1981.

[Lor63] Edward N. Lorenz. Deterministic nonperiodic flow. 20:130–141, 1963.

[Mey00] Carl Meyer. *Matrix analysis and applied linear algebra*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000. With 1 CD-ROM (Windows, Macintosh and UNIX) and a solutions manual (iv+171 pp.).

[MPP00] C. A. Morales, M. J. Pacifico, and E. R. Pujals. Strange attractors across the boundary of hyperbolic systems. *Comm. Math. Phys.*, 211(3):527–558, 2000.

[Rob81] Clark Robinson. Differentiability of the stable foliation for the model Lorenz equations. In *Dynamical systems and turbulence, Warwick 1980 (Coventry, 1979/1980)*, volume 898 of *Lecture Notes in Math.*, pages 302–315. Springer, Berlin, 1981.

[Rob84] Clark Robinson. Transitivity and invariant measures for the geometric model of the Lorenz equations. *Ergodic Theory Dynam. Systems*, 4(4):605–611, 1984.

[Rov93] Alvaro Rovella. The dynamics of perturbations of the contracting Lorenz attractor. *Bol. Soc. Brasil. Mat. (N.S.)*, 24(2):233–259, 1993.

[Ryc90] Marek Ryszard Rychlik. Lorenz attractors through Šil′nikov-type bifurcation. I. *Ergodic Theory Dynam. Systems*, 10(4):793–821, 1990.

[SS94] M. V. Shashkov and L. P. Shil′nikov. On the existence of a smooth invariant foliation in Lorenz-type mappings. *Differential Equations*, 30(4):536–544, 1994.

[Via00] Marcelo Viana. What's new on Lorenz strange attractors? *Math. Intelligencer*, 22(3):6–19, 2000.

[Vid14] José Vidarte. *Smooth perturbation of Lorenz-Like flow*. Ph.d thesis, ICMC-USP, http://www.teses.usp.br/teses/disponiveis/55/55135/tde-15072014-155326/en.php, april 2014.

[Wil79] R. F. Williams. The structure of Lorenz attractors. *Inst. Hautes Études Sci. Publ. Math.*, (50):73–99, 1979.

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